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GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND RELATED IDENTITIES

M. Tariq Rahim[*1], Laiba Hameed[2], Maqsood A. Khan[3]

[*1,2,3]*Department of Mathematics, Abbottabad University of Science & Technology, Abbottabad, Pakistan.*

[*1]tariqsms@gmail.com

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M. Tariq Rahim

Abstract
In this paper, we present new recurrence relations for a class of generalized Fibonacci polynomials and Lucas polynomial. We compute various properties for the sum of modified Generalized Fibonacci Polynomials. We present a number of formulas for the sum of these polynomials. Further, we investigate identities such as hypergeometric and integration for generalized Fibonacci polynomials.

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1. INTRODUCTION

Fibonacci numbers are important in nature, mathematics, and many other scientific domains because of their special characteristics and extensive uses. They are encountered in nature, where they affect spiral patterns in shells and galaxies, plant growth, and animal reproduction. They are also related to the golden ratio in number theory. They help predict market patterns in finance by using Fibonacci retracement, and in computer science they optimize data structures and search algorithms. These figures also create visually beautiful proportions in music, art, and architecture. Fibonacci numbers make a sequence in which each number is the sum of two preceding numbers.

Lucas numbers are important in number theory, combinatorics, and various scientific applications due to their unique properties and close relationship with Fibonacci numbers. They play a key role in primality testing, Diophantine equations, and cryptographic algorithms. In geometry, they appear in tiling problems and certain mathematical identities. Defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial values $L(0) = 2$ and $L(1) = 1$, the sequence follows "2, 1, 3, 4, 7, 11, 18, 29, ..." Lucas numbers also arise in electrical engineering and signal processing, highlighting their significance in both theoretical and practical fields.

The Fibonacci polynomials have significant

importance in mathematics. Many polynomials can be defined through recurrence relations similar to the Fibonacci sequence, and they generate Fibonacci numbers and the Fibonacci polynomial.

Fibonacci Sequence were initially introduced in Indian mathematics around 200 BCE when Pingala employed the Fibonacci sequence to compute the patterns of Sanskrit poetry made up of short and long syllables. Later, the Italian mathematician Leonàrdo of Pisa (also known as Fibonacci) popularized these sequences throughout Europe with his book "Liber Abàci," published in 1202. In1883, German mathematician E. Jàcobsthal and Belgian mathematician P.F. Byrd investigated the Fibonacci polynomial. Càtalan analysed the Fibonacci polynomials with recurrence relation

$$E_m(x) = xE_{m-1}(x) + E_{m-2}(x), \quad m \geq 3 \quad (1)$$

with $E_1(x) = 1, E_2(x) = x.$

P. F. Byrd studied the Fibonacci Polynomials described as follows:

$$B_m(x) = 2xB_{m-1}(x) + B_{m-2}(x), \quad m \geq 2 \quad (2)$$

with $B_0(x) = 0, B_1(x) = 1.$

Jàcobsthal analyzed the Fibonacci polynomials with recurrence relation

$$J_m(x) = J_{m-1}(x) + xJ_{m-2}(x), m \geq 3 \quad (3)$$

with $J_1(x) = J_2(x) = 1$

And

$$M_m(x) = xM_{m-1}(x) + M_{m-2}(x), \quad (4)$$

with $m \geq 2$ and $M_0(x) = 2, M_1(x) = x.$

Additionally, Hoggatt Jr. along with M. Bicknell discussed Fibonacci polynomial's generalization. They also provided a connection with Pascal's triangle diagonals. [8] From a geometric standpoint, S. Falcon and A. Plaza presented three-dimensional k-Fibonacci spirals [4]. They found the generalized Fibonacci sequence by recursively applying two geometrical alterations, including Pell sequences and ordinary Fibonacci sequences. They have shown many features and give an explicit definition of k-Fibonacci numbers [3]. Ayse-Nalli [5] investigated various for these Fibonacci numbers in addition to introducing the $h(n)$ -Fibonacci polynomial, which generalizes the Catalan and Byrd Fibonacci polynomials.

M. N. S. Swamy [2] investigated generalized Fibonacci polynomial identities.

Hoggèt together with Lind [12] makes identical symbolic substitutions of particular pat-terns within Fibonacci polynomials. Many polynomial problems have been published in the Fibonacci Quarterly. Fibonacci polynomials are among the additional identities that Hoggatt, Philips, and Leonard discovered [6]. C. Berg described the Fibonacci numbers and orthogonal polynomials as small q-Jacobi polynomials and verified analogous conclusions for the matrices [7].

K. Kàygisiz and A. Sàhin [11] derived the Binet formula and combinatorial representation for generalized order-k Lucas numbers using the characteristics of generalized Fibonacci numbers. In consideration of the Pascal Matrix, Gwang Yèon Lèe and Mustafa Asci [1] introduced (u, v) −Fibonacci polynomials. Weiping Wàng and Hui Wàng [9] defined (u, v) −Fibonacci polynomials $h_{u,v}(n)$ and the generalized (u, v) −Lucas polynomials $J_{u,v}(n)$, and introduced the generalization of Humbert polynomials $h_{u,v}^{(t)}(n)$ as the convolutions of $j_{u,v}(n)$.

Fibonacci polynomials were studied by Furkan Birola and Ozden Koruoglu [10] as complex hyperbolic functions. They looked at these polynomials' roots. Additionally, they provided some amazing identities for images of Fibonacci polynomial roots with a different Fibonacci polynomial class member. They discovered various excellent relationships between the modular group, Hecke groups, and generalized Hecke groups with geometric implications and the roots of Fibonacci polynomials.

Fibonacci polynomials have been investigated by numerous authors. In this paper, we present a new class of generalized Fibonacci and Lucas Polynomials by changing the recurrence relation while preserving initial terms.

2. Generalized Fibonacci Polynomials and Generalized Lucas Polynomials:

The generalized Fibonacci polynomials is expressed as:

$$\mathcal{F}_{n+1}(x) = \begin{cases} s & \text{if } n = 0 \\ sx^2 & \text{if } n = 1 \\ x^2\mathcal{F}_n(x) + \mathcal{F}_{n-1}(x) & \text{if } n \geq 2 \end{cases} \quad (5)$$

Here x is an indeterminate and s is a constant parameter, where both are taken from a commutative ring R (usually $R = \mathbb{R}$ or $\mathbb{C}$). Hence, the generalized Fibonacci polynomials $\mathcal{F}_n(x)$ form a sequence in the polynomial ring $R[x]$.

The Lucas and Fibonacci polynomials are known to be closely interrelated. Therefore, we present Lucas polynomial as;

$$\mathcal{L}_n(x) = \begin{cases} 2s & \text{if } n = 0 \\ sx^2 & \text{if } n = 1 \\ x^2\mathcal{L}_{n-1}(x) + \mathcal{L}_{n-2}(x) & \text{if } n \geq 2 \end{cases} \quad (6)$$

Where x and s have the same meaning as in the Fibonacci case.

The French mathematician Binet constructed two interesting formulas for the Fibonacci and Lucas numbers in the 19th century. In this scenario, Binet's formula can be used to describe modified generalized Fibonacci polynomials in functions of the roots of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of the following characteristic equation (7).

$$\mathfrak{R}^2 = x^2\mathfrak{R} + 1 \quad (7)$$

3. Properties of Generalized Fibonacci Polynomials

Fibonacci polynomials have a number of interesting properties that that result from their recurrence relations derivation. These properties are important in establishing and understanding its structure and provide a method for their use in various other fields.

Theorem 3.1. *The nth modified generalized Fibonacci polynomial has the form* $\mathcal{F}_n(x) = s\frac{\mathfrak{R}_1^n - \mathfrak{R}_2^n}{\mathfrak{R}_1 - \mathfrak{R}_2}$ (8) *where* $\mathfrak{R}_1 = \frac{x^2+\sqrt{x^2+4}}{2}$ *and* $\mathfrak{R}_2 = \frac{x^2-\sqrt{x^2+4}}{2}$ *are the roots of equation (7).*

Proof: We apply the induction on n. The result is obviously true for $n = 0$ and 1, i.e.,

$$\mathcal{F}_0(x) = s\frac{\mathfrak{R}_1^0 - \mathfrak{R}_2^0}{\mathfrak{R}_1 - \mathfrak{R}_2}$$
$$\mathcal{F}_0(x) = 0$$

And

$$\mathcal{F}_1(x) = s\frac{\mathfrak{R}_1^1 - \mathfrak{R}_2^1}{\mathfrak{R}_1 - \mathfrak{R}_2}$$
$$\mathcal{F}_1(x) = s$$

Assume that the result is true for i where $0 \leq i \leq z+1$, i.e.,

$$\mathcal{F}_i(x) = s\frac{\mathfrak{R}_1^i - \mathfrak{R}_2^i}{\mathfrak{R}_1 - \mathfrak{R}_2}$$

It comes from equation (8) and the definition of modified generalized Fibonacci polynomials. Since

$$\mathcal{F}_{n+1}(x) = x^2\mathcal{F}_n(x) + \mathcal{F}_{n-1}(x)$$

Substituting $n = z+1$, we have

$$\mathcal{F}_{z+2}(x) = x^2\mathcal{F}_{z+1}(x) + \mathcal{F}_z(x)$$
$$= x^2 s\frac{\mathfrak{R}_1^{z+1} - \mathfrak{R}_2^{z+1}}{\mathfrak{R}_1 - \mathfrak{R}_2} + \frac{\mathfrak{R}_1^z - \mathfrak{R}_2^z}{\mathfrak{R}_1 - \mathfrak{R}_2}$$
$$= s\left(\frac{x^2\mathfrak{R}_1^{z+1} + \mathfrak{R}_1^z - (x^2\mathfrak{R}_2^{z+1} + \mathfrak{R}_2^z)}{\mathfrak{R}_1 - \mathfrak{R}_2}\right)$$

Now substituting $\mathfrak{R}_1^{z+2} = x^2\mathfrak{R}_1^{z+1} + \mathfrak{R}_1^z$ and $\mathfrak{R}_2^{z+2} = x^2\mathfrak{R}_2^{z+1} + \mathfrak{R}_2^z$ we have

$$\mathcal{F}_{z+2}(x) = s\frac{\mathfrak{R}_1^{z+2} - \mathfrak{R}_2^{z+2}}{\mathfrak{R}_1 - \mathfrak{R}_2}$$

which proves the required result.

Theorem 3.2. *The nth modified generalized Lucas polynomials has the form*

$$\mathcal{L}_n(x) = s(\mathfrak{R}_1^n + \mathfrak{R}_2^n)$$

Where $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *has values exactly given in the previous theorem.*

Proof: The proof of this follows on the same lines as in Theorem 3.1.

Theorem 3.3. *For any integer* $n \geq 1$*, the following equalities holds:*

$$\begin{cases} \mathfrak{R}_1^{n+2} = x^2\mathfrak{R}_1^{n+1} + \mathfrak{R}_1^n \\ \mathfrak{R}_2^{n+2} = x^2\mathfrak{R}_2^{n+1} + \mathfrak{R}_2^n \end{cases} \quad (9)$$

Proof: Taking into account $\mathfrak{R}_1$ and $\mathfrak{R}_2$, the roots of the characteristic equation (7), we have

$\Re_1^2 = x^2\Re_1 + 1$ and $\Re_2^2 = x^2\Re_2 + 1$.

Multiplying both sides of the equations by $\Re_1^n$ and $\Re_2^n$, respectively, we have

$$\begin{cases}\Re_1^n\Re_1^2 = \Re_1^n(x^2\Re_1 + 1)\\ \Re_2^n\Re_2^2 = \Re_2^n(x^2\Re_2 + 1)\end{cases}$$

Which by simplification provides

$$\begin{cases}\Re_1^{n+2} = x^2\Re_1^{n+1} + \Re_1^n\\ \Re_2^{n+2} = x^2\Re_2^{n+1} + \Re_2^n\end{cases}$$

Theorem 3.4. *Let $n \geq 1$ be any integer. Then the following relation holds:*

$$s(\Re_1^n + \Re_2^n) = \mathcal{F}_{n+1}(x) + \mathcal{F}_n(x)$$

Proof: The right hand side can be found by considering $\Re_1 = \frac{-1}{\Re_2}$, and

$$\begin{aligned}\mathcal{F}_{n+1}(x) + \mathcal{F}_n(x) &= s\frac{\Re_1^{n+1} - \Re_2^{n+1}}{\Re_1 - \Re_2} + s\frac{\Re_1^{n-1} - \Re_2^{n-1}}{\Re_1 - \Re_2}\\ &= \frac{s}{\Re_1 - \Re_2}(\Re_1^{n+1} - \Re_2^{n+1} + \Re_1^{n-1} - \Re_2^{n-1})\\ &= s(\Re_1^n + \Re_2^n).\end{aligned}$$

Theorem 3.5. For every integer $\mathrm{n} \geq 0$, the modified generalized Fibonacci polynomials satisfy the following equalities:

$$(x^4 + 4)\mathcal{F}_{\mathrm{n}}^2(x) = \begin{cases}s^2((\Re_1^{\mathrm{n}} - \Re_2^n)^2 - 4(-1)^{\mathrm{n}}) & \text{if n is even,}\\ s^2((\Re_1^{\mathrm{n}} - \Re_2^{\mathrm{n}})^2 + 4(-1)^{\mathrm{n}}) & \text{if n is odd,}\end{cases}$$

where $\mathcal{L}_{\mathrm{n}}(x)$ are the generalized Lucas polynomials.

Proof: The proof follows from Binet's formula for $\mathrm{f}_{\mathrm{n}}(x)$ and $\mathcal{L}_{\mathrm{n}}(x)$. Let $\Re_1 = \frac{x^2+\sqrt{x^4+4}}{2}$ and $\Re_2 = \frac{x^2-\sqrt{x^4+4}}{2}$ be the roots of $\Re^2 = x^2\Re + 1$. Then:

$$\mathcal{F}_{\mathrm{n}}(x) = s\frac{\Re_1^{\mathrm{n}} - \Re_2^n}{\Re_1 - \Re_2}, \quad \mathcal{L}_{\mathrm{n}}(x) = s(\Re_1^n + \Re_2^{\mathrm{n}}).$$

Squaring $\mathrm{f}_{\mathrm{n}}(x)$ and substituting $\Re_1\Re_2 = -1$:

$$\begin{aligned}\mathcal{F}_{\mathrm{n}}^2(x) &= s^2\frac{(\Re_1^{\mathrm{n}} - \Re_2^{\mathrm{n}})^2}{(\Re_1 - \Re_2)^2}\\ &= s^2\frac{\Re_1^{2\mathrm{n}} + \Re_2^{2\mathrm{n}} - 2(-1)^{\mathrm{n}}}{\left(\frac{x^2+\sqrt{x^4+4}}{2} - \frac{x^2-\sqrt{x^4+4}}{2}\right)^2}\\ &= \frac{s^2}{x^2+4}(\Re_1^{2\mathrm{n}} + \Re_2^{2\mathrm{n}} - 2(-1)^{\mathrm{n}})\end{aligned}$$

By multiplying $(x^2 + 4)$ and adding $4s^2(-1)^{\mathrm{n}}$ on both sides, we get the following:

$(x^2 + 4)\mathrm{f}_{\mathrm{n}}^2(x) + 4s^2(-1)^{\mathrm{n}} = (\Re_1^{2\mathrm{n}}s^2 + \Re_2^{2\mathrm{n}}s^2 + 2(-1)$

By substituting $-1 = \Re_1\Re_2$ we get: $(x^2 + 4)\mathcal{F}_{\mathrm{n}}^2(x) + 4s^2(-1)^{\mathrm{n}} = s^2(\Re_1^{\mathrm{n}} + \Re_2^{\mathrm{n}})^2$

If n is even then: $(x^2 + 4)\mathrm{f}_n^2(x) + 4s^2 = s^2(\mathcal{L}_n(x)^2)$

If n is odd then: $(x^2 + 4)\mathrm{f}_{\mathrm{n}}^2(x) - 4s^2 = s^2(\mathcal{L}_{\mathrm{n}}(x)^2)$

4. Sums for Modified Versions of Generalized Fibonacci Polynomials

In this section, we compute various identities of the sums of modified generalized Fibonacci polynomials. This enables us to clearly and simply present a number of formulas for the sum of various polynomials.

Theorem 4.1. For fixed integers p and q with $0 \leq q \leq p - 1$, the following holds,

$$s\mathcal{F}_{p(p+2)}(x) = \mathcal{L}_p(x)\mathcal{F}_{p(n+1)+q}(x) - (-1)^p s\mathcal{F}_{pn+q}(x) \qquad (11)$$

Proof: Applying the Binet's formula of Lucas polynomials and Fibonacci polynomials, we have,

$$\mathcal{L}_p(x)\mathcal{F}_{p(n+1)+q}(x) = s\left(\Re_1^p + \Re_2^p\right)\left[s\frac{\Re_1^{p(n+1)+q} - \Re_2^{p(n+1)+q}}{\Re_1 - \Re_2}\right]$$
$$= \frac{s^2}{\Re_1 - \Re_2}\left[\Re_1^p\Re_1^{p(n+1)+q} - \Re_1^p\Re_2^{p(n+1)+q} + \Re_2^p\Re_1^{p(n+1)+q} - \Re_2^p\Re_2^{p(n+1)+q}\right]$$

Substituting $\Re_1^p = \left(-\frac{1}{\Re_2}\right)^p$ and $\Re_2^p = \left(-\frac{1}{\Re_1}\right)^p$, we get:

$$s\mathcal{F}_{p(p+2)}(x) = \mathcal{L}_p(x)\mathcal{F}_{p(n+1)+q}(x) - (-1)^p s\mathcal{F}_{pn+q}(x).$$

Theorem 4.2. For p, q with $0 \le q \le p-1$ the equality hold where p and q are fixed integers,

$$\sum_{n=0}^{\infty}\mathcal{F}_{pi+q}(x) = \frac{\mathcal{F}_{p(n+1)+q}(x) - \mathcal{F}_q(x) - (-1)^p\mathcal{F}_{pn+q}(x) - (-1)^q\mathcal{F}_{p-q}(x)}{\mathcal{L}_p(x) - (-1)^p - 1} \quad (12)$$

Proof: Applying the Binet's formula of new class of generalized Fibonacci Polynomials,

$$\sum_{n=0}^{n}\mathcal{F}_{pi+q}(x) = s\sum_{n=0}^{n}\frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2}$$
$$= \frac{s}{\Re_1 - \Re_2}\left(\sum_{n=0}^{n}\Re_1^{pi+q} - \sum_{n=0}^{n}\Re_2^{pi+q}\right)$$
$$= \frac{s}{\Re_1 - \Re_2}\left(\frac{\Re_1^{p(n+1)+q} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{p(n+1)+q} - \Re_2^q}{\Re_2^p - 1}\right)$$
$$= \frac{1}{\mathcal{L}_p(x) - (\Re_1^P + \Re_2^P) - 1}\left(\mathcal{F}_{p(n+1)+q}(x) - \mathcal{F}_q(x) - (-1)^p\mathcal{F}_{pn+q}(x) - (-1)^q\mathcal{F}_{p-q}(x)\right)$$

$$\sum_{n=0}^{n}\mathcal{F}_{pi+q}(x) = \frac{\mathcal{F}_{p(n+1)+q}(x) - \mathcal{F}_q(x) - (-1)^p\mathcal{F}_{pn+q}(x) - (-1)^q\mathcal{F}_{p-q}(x)}{\mathcal{L}_p(x) - (-1)^p - 1}.$$

Which proves the equality

Corollary 4.3. For odd $p, i.e., p = 2m+1$, we have

$$\sum_{i=0}^{n}\mathcal{F}_{(2m+1)i+q}(x) = \frac{\mathcal{F}_{(2m+1)(n+1)+q}(x) + \mathcal{F}_{(2m+1)n+q}(x) - \mathcal{F}_q(x) - (-1)^q\mathcal{F}_{(2m+1)-q}(x)}{\mathcal{L}_{2m+1}(x)} \quad (13)$$

Example 4.1. (1) If $m = 0$ then $p = 1$,

$$\sum_{i=0}^{n}\mathcal{F}_{i+q}(x) = \frac{\mathcal{F}_{n+q+1}(x) + \mathcal{F}_{n+q}(x) - \mathcal{F}_q(x) - (-1)^q\mathcal{F}_{1-q}(x)}{\mathcal{F}_1(x)} \quad (14)$$

(i) For $q = 0$:

$$\sum_{i=0}^{n}\mathcal{F}_i(x) = \frac{\mathcal{F}_{n+1}(x) + \mathcal{F}_n(x) - s}{sx^2}$$

(2) If $m = 1$ then $p = 3$

$$\sum_{i=0}^{n} \mathcal{F}_{3i+q}(x) = \frac{\mathcal{F}_{3n+q+3}(x) + \mathcal{F}_{3n+q}(x) - \mathcal{F}_q(x) - (-1)^q \mathcal{F}_{3-q}(x)}{sx^6 + 3sx^2} \quad (15)$$

(i) For $q = 0$:

$$\sum_{i=0}^{n} \mathcal{F}_{3i}(x) = \frac{\mathcal{F}_{3n+3}(x) + \mathcal{F}_{3n}(x) - sx^4 - s}{sx^6 + 3sx^2}$$

(ii) For $q = 1$:

$$\sum_{i=0}^{n} \mathcal{F}_{3i+1}(x) = \frac{\mathcal{F}_{3n+4}(x) + \mathcal{F}_{3n+1}(x) - s + sx^2}{sx^6 + 3sx^2}$$

(iii) For $q = 2$:

$$\sum_{i=0}^{n} \mathcal{F}_{3i+2}(x) = \frac{\mathcal{F}_{3n+5}(x) + \mathcal{F}_{3n+2}(x) - sx^2 - s}{sx^6 + 3sx^2}$$

(3) If $m = 2$ then $p = 5$,

$$\sum_{i=0}^{n} \mathcal{F}_{5i+q}(x) = \frac{\mathcal{F}_{5n+q+5}(x) + \mathcal{F}_{5n+q}(x) - \mathcal{F}_q(x) - (-1)^q \mathcal{F}_{5-q}(x)}{sx^{10} + 5sx^6 + 5sx^2} \quad (16)$$

(i) For $q = 0$:

$$\sum_{i=0}^{n} \mathcal{F}_{5i}(x) = \frac{\mathcal{F}_{5n+5}(x) + \mathcal{F}_{5n}(x) - sx^8 - 3sx^4 - s}{sx^{10} + 5sx^6 + 5sx^2}$$

(ii) For $q = 1$:

$$\sum_{i=0}^{n} \mathcal{F}_{5i+1}(x) = \frac{\mathcal{F}_{5n+6}(x) + \mathcal{F}_{5n+1}(x) + sx^6 + 2sx^2 - s}{sx^{10} + 5sx^6 + 5sx^2}$$

(iii) For $q = 2$:

$$\sum_{i=0}^{n} \mathcal{F}_{5i+2}(x) = \frac{\mathcal{F}_{5n+7}(x) + \mathcal{F}_{5n+2}(x) - sx^4 - sx^2 - s}{sx^{10} + 5sx^6 + 5sx^2}$$

(iv) For $q = 3$:

$$\sum_{i=0}^{n} \mathrm{f}_{5i+3}(x) = \frac{\mathcal{F}_{5n+8}(x) + \mathcal{F}_{5n+3}(x) - sx^4 + sx^2 - s}{sx^{10} + 5sx^6 + 5sx^2}$$

(v) For $q = 4$:

$$\sum_{i=0}^{n} \mathcal{F}_{5i+4}(x) = \frac{\mathcal{F}_{5n+9}(x) + \mathcal{F}_{5n+4}(x) - sx^6 - 2sx^2 - s}{sx^{10} + 5sx^6 + 5sx^2}.$$

Corollary 4.4. For $p = 2m$, Equation(14) becomes:

$$\sum_{i=0}^{n} \mathcal{F}_{2mi+q}(x) = \frac{\mathcal{F}_{2mn+2m+q}(x) - (-1)^{2m}\mathcal{F}_{2mn+q}(x) - \mathcal{F}_q(x) - (-1)^q \mathcal{F}_{2m-q}(x)}{\mathrm{l}_{2m}(x) - 2}. \quad (17)$$

Example 4.2. (1) If $m = 1$ then $p = 2$

$$\sum_{i=0}^{n} \mathcal{F}_{2i+q}(x) = \frac{\mathcal{F}_{2n+2+q}(x) - \mathcal{F}_{2n+q}(x) - \mathcal{F}_q(x) - (-1)^q \mathcal{F}_{2-q}(x)}{sx^4 + 2s - 2} \quad (18)$$

(i) For $q = 0$:

$$\sum_{i=0}^{n} \mathcal{F}_{2i}(x) = \frac{\mathrm{f}_{2n+2}(x) - \mathcal{F}_{2n}(x) - sx^2}{sx^4 + 2s - 2}$$

(ii) For $q = 1$:

$$\sum_{i=0}^{n} \mathcal{F}_{2i+1}(x) = \frac{\mathcal{F}_{2n+3}(x) - \mathcal{F}_{2n+1}(x)}{sx^4 + 2s - 2}$$

(iii) For $q = 2$:

$$\sum_{i=0}^{n} \mathcal{F}_{2i+2}(x) = \frac{\mathcal{F}_{2n+4}(x) - \mathcal{F}_{2n+2}(x) - sx^2}{sx^4 + 2s - 2}$$

(2) If $m = 2$ then $p = 4$

$$\sum_{i=0}^{n} \mathcal{F}_{4i+q}(x) = \frac{\mathcal{F}_{4n+4+q}(x) - \mathcal{F}_{4n+q}(x) - \mathcal{F}_q(x) - (-1)^q \mathcal{F}_{4-q}(x)}{sx^8 + 4sx^4 + 2s - 2} \quad (19)$$

(i) For $q = 0$

$$\sum_{i=0}^{n} \mathcal{F}_{4i}(x) = \frac{\mathcal{F}_{4n+4}(x) - \mathcal{F}_{4n}(x) - sx^6 - 2sx^2}{sx^8 + 4sx^4 + 2s - 2}$$

(ii) For $q = 1$

$$\sum_{i=0}^{n} \mathcal{F}_{4i+1}(x) = \frac{\mathcal{F}_{4n+5}(x) - \mathcal{F}_{4n+1}(x) + sx^4}{sx^8 + 4sx^4 + 2s - 2}$$

(iii) For $q = 2$

$$\sum_{i=0}^{n} \mathcal{F}_{4i+2}(x) = \frac{\mathcal{F}_{4n+6}(x) - \mathcal{F}_{4n+2}(x) - 2sx^2}{sx^8 + 4sx^4 + 2s - 2}$$

(iv) For $q = 3$

$$\sum_{i=0}^{n} \mathcal{F}_{4i+2}(x) = \frac{\mathcal{F}_{4n+7}(x) - \mathcal{F}_{4n+3}(x) - sx^4}{sx^8 + 4sx^4 + 2s - 2}$$

(v) For $q = 4$

$$\sum_{i=0}^{n} \mathcal{F}_{4i+4}(x) = \frac{\mathcal{F}_{4n+8}(x) - \mathcal{F}_{4n+4}(x) - sx^6 - 2sx^2}{sx^8 + 4sx^4 + 2s - 2}.$$

5. Identities of Confluent Hypergeometrics for modified Generalized Fibonacci Polynomials

Generating function for modified Fibonacci polynomials and Lucas polynomials have been derived in this section. On the basis of the generating function some characteristics of modified generalized Fibonacci polynomials are presented using confluent Hypergeometric functions. Several identities using special functions and simple algebra are provided.

5.1 Derivation of Generating Function for Generalized Fibonacci Polynomials

We begin by recalling the recurrence relation of generalized Fibonacci polynomials:

$$\mathcal{F}_n = x^2\mathcal{F}_{n-1} + \mathcal{F}_{n-2} \quad \text{with} \quad \mathcal{F}_0 = 0 \quad \text{and} \quad \mathcal{F}_1 = s$$

Multiplying t^n on both sides and take summation from 2 to ∞ on both sides,

$$\sum_{n=2}^{\infty} \mathcal{F}_n\, t^n = x^2 \sum_{n=2}^{\infty} \mathcal{F}_{n-1}\, t^n + \sum_{n=2}^{\infty} \mathcal{F}_{n-2}\, t^n \qquad (20)$$

Simplifying the right hand side,

$$\text{Let} \quad m = n-1 \quad \text{for} \quad x^2 \sum_{n=2}^{\infty} \mathcal{F}_{n-1}\, t^n$$

So, for $n = 2 \implies m = 1$ and for $n = \infty \implies m = \infty$

Therefore,

$$x^2 \sum_{n=2}^{\infty} \mathcal{F}_{n-1}\, t^n = x^2 t\mathcal{F}(t) \quad (21)$$

$$\text{Let} \quad m = n-2 \quad \text{for} \quad \sum_{n=2}^{\infty} \mathcal{F}_{n-2}\, t^n$$

So, for $n = 2 \implies m = 0$ and for $n = \infty \implies m = \infty$

Therefore,

$$\sum_{n=2}^{\infty} \mathcal{F}_{n-2}\, t^n = t^2\mathcal{F}(t) \quad (22)$$

By substituting equations (21) and (22) in equation (20) we get:

$$\mathcal{F}(t) = \frac{st}{(1 - x^2t - t^2)} \qquad (23)$$

Hence the above equation (23) is the Generating function of modified generalized Fibonacci polynomials.

Theorem 5.1 If $\mathcal{F}_n(x)$ is the modified generalized Fibonacci polynomial, then the following identity holds:

$$\sum_{n=0}^{\infty} \mathcal{F}_n\,(x) \frac{t^{n-1}}{n!} = se^{x^2t} \cdot {}_1\mathcal{F}_0(n+1; -; t^2).$$

Proof. Since the generating function of the modified generalized Fibonacci Polynomials is:

$$\sum_{n=0}^{\infty} \mathcal{F}_n\,(x)t^n = \frac{st}{1 - x^2t - t^2}$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n\,(x)t^{n-1} == \frac{s}{1 - x^2t - t^2}$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n\,(x)t^{n-1} == s \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} x^{2(n-r)}\, t^r\, ({}^nC_r)$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)t^{n-1} = s\sum_{n=0}^{\infty}\sum_{r=0}^{n} c^{2n-2r}\, t^{n+r}\, \frac{n!}{r!\,(n-r)!}$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)t^{n-1} = s\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} x^{2n+2r-2r}\, t^{(n+r)+r}\, \frac{(n+r)!}{r!\,n!}$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)t^{n-1} = s\sum_{n=0}^{\infty} \frac{(x^2t)^n}{n!}\sum_{r=0}^{\infty} \frac{t^{2r}}{r!}(n+r)!$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)t^{n-1} = s\sum_{n=0}^{\infty} \frac{(x^2t)^n}{n!}\sum_{r=0}^{\infty} \frac{t^{2r}}{r!}n!\,(n+1)_r$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)\frac{t^{n-1}}{n!} = se^{x^2t}\sum_{r=0}^{\infty} \frac{t^{2r}}{r!}(n+1)_r$$

$$\sum_{n=0}^{\infty} \mathcal{F}_n(x)\frac{t^{n-1}}{n!} = se_1^{x^2t}F_0(n+1,-,t^2).$$

Which proves the required result.

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5.2 Derivation of generating function for modified Generalized Lucas polynomials

We begin by recalling the recurrence relation of modified generalized Lucas polynomials:

$$\mathcal{L}_n = x^2\mathcal{L}_{n-1} + \mathcal{L}_{n-2} \quad \text{with} \quad \mathcal{L}_0 = 2s \quad \text{and} \quad \mathcal{L}_1 = sx^2$$

Multiplying t^n on both sides and taking the summation from 2 to ∞ on both sides,

$$\sum_{n=2}^{\infty} \mathcal{L}_n\, t^n = x^2\sum_{n=2}^{\infty} \mathcal{L}_{n-1}\, t^n + \sum_{n=2}^{\infty} \mathcal{L}_{n-2}\, t^n \qquad (24)$$

Simplifying the right hand side,

$$\text{Let} \quad m = n-1 \quad \text{for} \quad x^2\sum_{n=2}^{\infty} \mathcal{L}_{n-1}\, t^n$$

So, for $n = 2 \implies m = 1$ and for $n = \infty \implies m = \infty$

Therefore,

$$x^2\sum_{n=2}^{\infty} \mathcal{F}_{n-1}\, t^n = x^2t\mathcal{L}(t) - 2x^2st \qquad (25)$$

$$\text{Let} \quad m = n-2 \quad \text{for} \quad \sum_{n=2}^{\infty} \mathcal{L}_{n-2}\, t^n$$

So, for $n = 2 \implies m = 0$ and for $n = \infty \implies m = \infty$
Therefore,

$$\sum_{n=2}^{\infty} \mathcal{L}_{n-2}\, t^n = t^2 \mathcal{L}(t) \qquad (26)$$

Substituting eq. (27) and eq. (28) in eq, (26)

$$\mathcal{L}(t) = \frac{s\Delta}{(1 - x^2 t - t^2)} \qquad (27)$$

Hence the above eq. (29) is the Generating function of modified generalized Lucas polynomials.
Theorem 5.2 If $\mathcal{L}_n(x)$ is the modified generalized Lucas polynomial, then the following identity holds:

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) \frac{t^n}{n!} = \Delta s e^{x^2 t}\, {}_1\mathcal{F}_0(n + 1 - t^2) \qquad (28)$$

Proof. Since the generating function for the modified generalized Lucas polynomials is given by:

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \frac{\Delta s}{1 - x^2 t - t^2}.$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} (x^2 t + t^2)^n$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} t^n (x^2 + t)^n$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} x^{2(n-r)}\, t^r\, ({}^nC_r)$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} \sum_{r=0}^{n} x^{2n-2r}\, t^{n+r} \frac{n!}{r!\,(n-r)!}$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} x^{2n+2r-2r}\, t^{(n+r)+r} \frac{(n+r)!}{r!\,n!}$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} \frac{(x^2 t)^n}{n!} \sum_{r=0}^{\infty} \frac{t^{2r}}{r!} (n+r)!$$

$$\sum_{n=0}^{\infty} \mathcal{L}_n(x) t^n = \Delta s \sum_{n=0}^{\infty} \frac{(x^2 t)^n}{n!} \sum_{r=0}^{\infty} \frac{t^{2r}}{r!} n!\,(n+1)_r$$

$$\sum_{n=0}^{\infty} \mathcal{L}(x) \frac{t^n}{n!} = \Delta s e^{x^2 t}\, {}_1F_0(n + 1, -, t^2).$$

Which proves the required identity.

Theorem 5.3. For modified generalized Fibonacci $\mathcal{F}_n$ and Lucas $\mathcal{L}_n$ polynomials the following equality hold:

$$\mathcal{L}_n(x) = 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) \qquad (29)$$

Proof: Taking the right hand side and applying Binet's formula:

$$\begin{aligned} 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) &= 2\left(s\frac{\mathfrak{R}_1^{n+1} - \mathfrak{R}_2^{n+1}}{\mathfrak{R}_1 - \mathfrak{R}_2}\right) - x^2\left(s\frac{\mathfrak{R}_1^{n} - \mathfrak{R}_2^{n}}{\mathfrak{R}_1 - \mathfrak{R}_2}\right) \\ 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) &= \frac{s}{\mathfrak{R}_1 - \mathfrak{R}_2}\left(\mathfrak{R}_1^{n}(2\mathfrak{R}_1 - x^2) - \mathfrak{R}_2^{n}(2\mathfrak{R}_2 - x^2)\right) \\ 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) &= \frac{s}{\mathfrak{R}_1 - \mathfrak{R}_2}(\mathfrak{R}_1 - \mathfrak{R}_2)(\mathfrak{R}_1^{n} + R_2^{n}) \\ 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) &= s(\mathfrak{R}_1^{n} + R_2^{n}) \\ 2\mathcal{F}_{n+1}(x) - x^2\mathcal{F}_n(x) &= \mathcal{L}_{\mathrm{n}}(\mathrm{x}) \end{aligned}$$

Which proves the desired result.

Theorem 5.4. For generalized Fibonacci $\mathcal{F}_n$ and Lucas $\mathcal{L}_n$ polynomials the following equality hold:

$$\mathcal{L}_{n+1}(x) = x\mathcal{F}_{n+1}(x) + 2x\mathcal{F}_n(x) \qquad (30)$$

Proof: Using Recurrence relation for $\mathcal{F}_{n+1}(x)$ and substituting equation (31) the proof is clear.

Theorem 5.5. For $\mathcal{F}(x)$ and $\mathcal{L}(x)$ the following equalities hold:

$$\int_{-1}^{1} \mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(x)dx = 0 \qquad (31)$$

Proof: To prove the desired result, we have to show that $\mathrm{f}_{n+1}(x)\mathrm{l}_{n+1}(x)$ is odd.

Recall the parity properties:

$$\mathcal{F}_m(-x) = (-1)^{m+1}\mathcal{F}_m(x)$$
$$\mathcal{F}_m(-x) = (-1)^{m+1}\mathcal{F}_m(x)$$

For $m = n + 1$

$$\mathcal{F}_{n+1}(-x) = (-1)^{n+2}\mathcal{F}_{n+1}(x) = (-1)^{n}\mathcal{F}_{n+1}(x)$$
$$\mathcal{L}_{n+1}(-x) = (-1)^{n+1}\mathcal{L}_{n+1}(x)$$

So, the product is:

$$\begin{aligned} \mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(x) &= [(-1)^{n}\mathcal{F}_{\mathrm{n}+1}(x)][(-1)^{n+1}\mathcal{L}_{n+1}(x)] \\ \mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(x) &= -\mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(c) \end{aligned}$$

Here, $\mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(x)$ is odd. Taking Integration over symmetric limits:

$$\int_{-1}^{1} (odd\ function)\, dc = 0$$
$$\int_{-1}^{1} \mathcal{F}_{n+1}(x)\mathcal{L}_{n+1}(x)dx = 0$$

Which is the required result.

Theorem 5.6. For $\mathcal{F}(x)$ and $\mathcal{L}(x)$ the following equalities hold:

$$\int 2x.\mathcal{F}_{n+1}(x)e^{\mathcal{L}_{n+1}(x)}dx = \frac{e^{\mathcal{L}_{n+1}(x)}}{n+1} + c, \quad \text{where } c \text{ is constant.} \qquad (32)$$

Proof: In order to prove equation (34), we take the derivative of the right hand side with respect to x:

$$\frac{d}{dx}\left(\frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}+c\right) = \frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}\mathcal{L}'_{n+1}(x)$$

$$\frac{d}{dx}\left(\frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}+c\right) = \frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}(n+1)\left(2x\mathcal{F}_{n+1}(x)\right)$$

$$\frac{d}{dx}\left(\frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}+c\right) = 2x\mathcal{F}_{n+1}(x)e^{\mathcal{L}_{n+1}(x)}$$

$$\int \frac{d}{dx}\left(\frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}+c\right)dx = \int 2x\mathcal{F}_{n+1}(x)e^{\mathcal{L}_{n+1}(x)}dx$$

$$\int 2x\mathcal{F}_{n+1}(x)e^{\mathcal{L}_{n+1}(x)}dx = \frac{e^{\mathcal{L}_{n+1}(x)}}{n+1}+c.$$

Hence the desired result is proved.

Corollary 5.7. The sum of two consecutive modified generalized Fibonacci polynomials is:

$$\sum_{n}^{\infty}[\mathcal{F}_{\mathrm{n}}(\mathrm{x})+\mathcal{F}_{\mathrm{n+1}}(\mathrm{x})]\,t^n = \frac{s(1+t)}{1-x^2t-t^2} \qquad (33)$$

Proof: Applying the generating function of modified Fibonacci polynomials on the right hand side,

$$\sum_{n}^{\infty}[\mathcal{F}_{\mathrm{n}}(\mathrm{x})+\mathcal{F}_{\mathrm{n+1}}(\mathrm{x})]\,t^n = \frac{st}{(1-x^2t-t^2)}+\frac{s}{(1-x^2t-t^2)}$$

$$\sum_{n}^{\infty}[\mathcal{F}_{\mathrm{n}}(\mathrm{x})+\mathcal{F}_{\mathrm{n+1}}(\mathrm{x})]\,t^n = \frac{s(1+t)}{1-x^2t-t^2}.$$

Corollary 5.8. The sum of two consecutive modified generalized Lucas polynomials is:

$$\sum_{n}^{\infty}[\mathcal{L}_{\mathrm{n}}(\mathrm{x})+\mathcal{L}_{\mathrm{n+1}}(\mathrm{x})]\,t^n = \frac{s(2+x^2)+st(2-x^2)}{(1-x^2t-t^2)} \qquad (34)$$

Proof: Applying the generating function of modified Lucas polynomials on the left hand side:

$$\sum_{n=0}^{\infty}[\mathcal{L}_n(x)+\mathcal{L}_{n+1}(x)]\,t^n = \frac{s(2-x^2t)}{(1-x^2t-t^2)}+\frac{s(x^2+2t)}{(1-x^2t-t^2)}$$

$$\sum_{n=0}^{\infty}[\mathcal{L}_n(x)+\mathcal{L}_{n+1}(x)]\,t^n = \frac{2s-sx^2t+sx^2+2st}{(1-x^2t-t^2)}$$

$$\sum_{n=0}^{\infty}[\mathcal{L}_n(x)+\mathcal{L}_{n+1}(x)]\,t^n = \frac{s(2+x^2)+st(2-x^2)}{(1-x^2t-t^2)}.$$

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