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NEW GENERALIZED OSTROWSKI-TYPE INEQUALITIES INVOLVING INTEGRAL MEANS OVER END INTERVALS

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Abstract
In this paper, we establish a new Ostrowski-type inequality involving integral means over end intervals by introducing a generalized Montgomery-type identity and a generalized Peano kernel. The obtained inequality provides an explicit estimate for twice differentiable functions under the boundedness assumption on the second derivative. Furthermore, some important special cases are discussed to illustrate the applicability of the proposed result. The developed inequality extends and generalizes existing Ostrowski-type inequalities and may be useful in numerical integration, approximation theory, and error estimation.

Introduction

Integral inequalities play an important role in mathematical analysis and its applications. Among them, Ostrowski's inequality is one of the most significant results because it gives an estimate for the deviation of a differentiable function from its integral mean. Since its introduction by Ostrowski in 1938,[1] this inequality has been widely studied and extended in different directions due to its applications in numerical integration, approximation theory, quadrature formulas, and error estimation. [2, 3]

The classical Ostrowski inequality has attracted considerable attention from many researchers.[1, 4, 5] Numerous extensions, refinements, and generalizations have been established under different assumptions on the derivatives of functions. In particular, inequalities involving convex functions, bounded variation, Lipschitz conditions, Peano kernels, and Montgomery identities have been investigated extensively. [1]These developments have significantly enhanced the applicability of Ostrowski-type inequalities in both theoretical and applied mathematics.[6]

Among the various approaches available in the literature, the use of Montgomery identities and generalized Peano kernels has proved to be an effective technique for deriving sharp integral inequalities. [4, 5, 7]These methods provide elegant representations of the deviation of a function from its integral mean and lead to improved estimates under different smoothness assumptions. Consequently, they have become an important tool in the study of Ostrowski-type inequalities and related approximation results [8, 9] .

Preliminaries

In this section, we recall some basic definitions and well-known results that will be used throughout the paper.

Let $f: [a, b] \to \mathbb{R}$ be a real-valued function defined on a finite interval $[a, b]$.

Definition 2.1 (Integral Mean)

The integral mean of a function f over the interval $[a, b]$ is defined by

$$M(f; a, b) = \frac{1}{(b-a)} \int_a^b f(t)dt$$

More generally, for any subinterval $[c, d] \subseteq [a, b]$,

$$M(f; c, d) = \frac{1}{(d-c)} \int c^{d} f(t)dt$$

Integral means play a fundamental role in the theory of Ostrowski-type inequalities and numerical quadrature formulas.

Definition 2.2 (Lp Spaces)

For $1 \leq p < \infty$, the space $Lp[a, b]$ consists of all measurable functions satisfying

$$||f||_p = \left[\int_a^b |f(t)|^p dt\right]^{\frac{1}{p}} < \infty$$

$$Forp = \infty,$$

$$||f||\infty = \sup\{|f(t)|: t \in [a, b]\}$$

These norms will be used to derive error bounds for generalized Ostrowski-type inequalities.

Definition 2.3 (Conjugate Exponents)

Let $p > 1$ and $q > 1$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then p and q are called conjugate exponents.

Lemma 2.1 (Holder's Inequality)

Let $u \in Lp[a, b]$ and $v \in Lq[a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\int_a^b |u(t)v(t)|dt$$

$$\leq [\int_a^b |u(t)|^p dt]^{\frac{1}{p}} \left[\int_a^b |v(t)|^q dt\right]^{\frac{1}{q}}$$

Proof.

The proof follows directly from the classical Holder inequality and may be found in standard texts on real analysis and integral inequalities.

Definition 2.4 (Beta Function)
The Beta function is defined by

$$\boldsymbol{B(m,n)=\int_0^1 t^{(m-1)}\ (1-t)^{(n-1)}dt,\quad m,n > 0}$$

The Beta function frequently appears in the evaluation of kernel integrals arising in Ostrowski-type inequalities.

Classical Ostrowski Inequality

For completeness, we recall the classical Ostrowski inequality [2].

Let $\boldsymbol{f}:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ be differentiable on $(\boldsymbol{a},\boldsymbol{b})$ and suppose that

$$\left|\boldsymbol{f}'(\boldsymbol{t})\right|\leq \boldsymbol{M},\quad \boldsymbol{t}\in[\boldsymbol{a},\boldsymbol{b}].$$

Then

$$\left|\boldsymbol{f(x)-\frac{1}{b-a}\int_a^b f(t)dt}\right| \leq \left[\frac{\boldsymbol{1}}{\boldsymbol{4}}+\frac{\left(\boldsymbol{x}-\frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right)^{\boldsymbol{2}}}{(\boldsymbol{b-a})^{\boldsymbol{2}}}\right](\boldsymbol{b-a})\boldsymbol{M}\quad \boldsymbol{for\ every\ x}\in[\boldsymbol{a},\boldsymbol{b}]$$

The above inequality serves as the foundation for numerous generalizations and extensions available in the literature.

The results established in the next section are obtained through the construction of a generalized Peano kernel and a new Montgomery-type identity involving integral means over end intervals.

Main Identity

In this section, we establish a new Montgomery-type identity involving integral means over end intervals [10] . The identity is obtained through the construction of a generalized Peano kernel and serves as the principal tool for deriving the main Ostrowski-type inequalities presented in this paper. Several special cases of the identity are also discussed.

Lemma 3.1

Let $f:[a,b]\to\mathbb{R}$ be a continuous mapping. Define the generalized Peano kernel. $p(x,t)$ by For $t\in[a,x]$

$$p(x,t)=\begin{cases}\dfrac{\alpha}{\alpha+\beta}\dfrac{1}{x-a}\left[t-\left(\dfrac{a+x}{2}\right)^2\right](x-t), & if\ \ t\in[a,x]\\[2ex] \dfrac{\beta}{\alpha+\beta}\dfrac{1}{b-x}\left[t-\left(\dfrac{b+x}{2}\right)^2\right](x-t), & if\ t\in(x,b]\end{cases}$$

Then the following Montgomery-type identity holds:

$$\begin{aligned}\int_a^b p(x,t)\ddot{f}(t)dt &= \frac{1}{\alpha+\beta}\left[\frac{-\alpha}{x-a}\left\{-x+\left(\frac{a+x}{2}\right)^2\right\}+\frac{\beta}{b-x}\left\{-x+\left(\frac{b+x}{2}\right)^2\right\}\right]f(x)\\ &+\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-2a+\left(\frac{a+x}{2}\right)^2\right\}f(a)-\frac{\beta}{b-x}\left\{x-2b+\left(\frac{b+x}{2}\right)^2\right\}f(b)\right]\\ &-\frac{1}{\alpha+\beta}\left[\alpha\left\{a-\left(\frac{a+x}{2}\right)^2\right\}\dot{f}(a)+\beta\left\{b-\left(\frac{b+x}{2}\right)^2\right\}\dot{f}(b)\right]\\ &-\frac{2}{\alpha+\beta}\left[\alpha M(f:a,x)+\beta M(f:x,b)\right].\end{aligned}$$

Proof

Splitting the integral into two parts over the intervals $[\boldsymbol{a},\boldsymbol{x}]$, $[\boldsymbol{x},\boldsymbol{b}]$ and applying integration by parts to each term, we obtain the desired representation. Simplification of the resulting expressions completes the proof. ▮

The identity established in Lemma 3.1 provides a useful representation involving integral means over end intervals. This identity will be employed in the next section to derive several new Ostrowski-type inequalities under different assumptions on the second derivative of the function.

Main Results

4.1 Introduction

In this section, we derive new Ostrowski-type inequalities involving integral means over end intervals. The results are obtained by applying the Montgomery-type identity established in Lemma 3.1 together with suitable estimates of the generalized Peano kernel. The obtained inequalities provide bounds for the deviation of a function from its integral means in terms of the second derivative. Furthermore, several special cases are discussed to demonstrate the effectiveness of the derived results.

Theorem 4.1

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping and given below

$$\tau(x,\alpha,\beta) := \frac{1}{\alpha+\beta}\left[\frac{-\alpha}{x-a}\left\{-x+\left(\frac{a+x}{2}\right)^2\right\} + \frac{\beta}{b-x}\{-x + \left(\frac{b+x}{2}\right)^2\}\right] f(x) + \frac{1}{\alpha+\beta}[\frac{\alpha}{x-a}\{x-2a + \left(\frac{a+x}{2}\right)^2\} f(a) - \frac{\beta}{b-x}\{x-2b + \left(\frac{b+x}{2}\right)^2\} f(b)\right] - \frac{1}{\alpha+\beta}[\alpha\{a - \left(\frac{a+x}{2}\right)^2\} \dot{f}(a) + \beta\left\{b-\left(\frac{b+x}{2}\right)^2\right\}\dot{f}(b)\right] - \frac{2}{\alpha+\beta}[\alpha M(f: a,x) + \beta M(f:x,b)].$$

Where $M(f:a,b)$ is the integral mean. Then

$$|\tau(x,\alpha,\beta)| \le \frac{\|\ddot{f}\|_\infty}{2(\alpha+\beta)}[\alpha\left\{(x-a)\left(a-\left(\frac{a+x}{2}\right)^2\right) + \frac{(x-a)^2}{3} - \beta\left\{(b-x)\left(b-\left(\frac{b+x}{2}\right)^2\right) - \frac{(b-x)^2}{3}\right\}\right] \qquad Equation(4.1)$$

Where $\ddot{f} \in l_{\infty\,[a,b]}$

Proof.

$$\left|\int_a^b p(x,t)\ddot{f}(t)dt\right| \le \int_a^b |p(x,t)||\ddot{f}(t)|dt = \|\ddot{f}\|_\infty \int_a^b |p(x,t)|dt \quad Equation\ (4.2)$$

Now solving integral

$$\int_a^b |p(x,t)|dt = \int_a^x |p(x,t)|dt + \int_x^b |p(x,t)|dt = I_1 + I_2$$

$$I_1 = \frac{\alpha}{\alpha+\beta}\int_a^x \frac{t-\left(\frac{a+x}{2}\right)^2}{x-a}(x-t)dt$$

$$= \frac{\alpha}{(\alpha+\beta)(x-a)}\int_a^x \left(t-\left(\frac{a+x}{2}\right)^2\right)(x-t)dt$$

$$= \frac{\alpha}{(\alpha+\beta)(x-a)}[\frac{-(x-t)^2}{2}\left(t-\left(\frac{a+x}{2}\right)^2\right)\Bigg|_a^x - \frac{(x-t)^3}{6}\Bigg|_a^x$$

$$= \frac{\alpha}{(\alpha+\beta)(x-a)}\left[\frac{(x-a)^2}{2}\left(a-\left(\frac{a+x}{2}\right)^2\right) + \frac{(x-a)^3}{6}\right]$$

Hence

$$I_1 = \int_a^x |p(x,t)|dt = \frac{\alpha}{2(\alpha+\beta)}\left[(x-a)\left(a-\left(\frac{a+x}{2}\right)^2\right)+\frac{(x-a)^2}{3}\right]$$

Now we take

$$I_2 = \int_x^b |p(x,t)|dt = \frac{\beta}{(\alpha+\beta)(b-x)}\int_a^x \left(t-\left(\frac{b+x}{2}\right)^2\right)(x-t)dt$$

$$= \frac{\beta}{(\alpha+\beta)(b-x)}\left[\frac{-(x-t)^2}{2}\left(t-\left(\frac{b+x}{2}\right)^2\right)\Big|_x^b - \frac{(x-t)^3}{6}\Big|_x^b\right]$$

$$= \frac{\beta}{(\alpha+\beta)(b-x)}\left[\frac{-(x-b)^2}{2}\left(b-\left(\frac{b+x}{2}\right)^2\right)-\frac{(x-b)^3}{6}\right]$$

Hence

$$I_2 = \int_x^b |p(x,t)|dt = \frac{-\beta}{2(\alpha+\beta)}\left[(b-x)\left(b-\left(\frac{b+x}{2}\right)^2\right)-\frac{(b-x)^2}{3}\right]$$

Now adding I_1 and I_2

$$\int_a^b |p(x,t)|dt = \frac{\alpha}{2(\alpha+\beta)}\left[(x-a)\left(a-\left(\frac{a+x}{2}\right)^2\right)+\frac{(x-a)^2}{3}\right] - \frac{\beta}{2(\alpha+\beta)}\left[(b-x)\left(b-\left(\frac{b+x}{2}\right)^2\right)-\frac{(b-x)^2}{3}\right] \qquad Equation\ (4.3)$$

Using Equation (4.3) in Equation (4.2) we get required Equation (4.1)

$$\left|\int_a^b p(x,t)\ddot{f}\ (t)dt\right| \le \frac{\|\ddot{f}\|_\infty}{2(\alpha+\beta)}\left[\alpha\left\{(x-a)\left(a-\left(\frac{a+x}{2}\right)^2\right)+\frac{(x-a)^2}{3}\right\} - \beta\left\{(b-x)\left(b-\left(\frac{b+x}{2}\right)^2\right)-\frac{(b-x)^2}{3}\right\}\right]$$

Hence Proved.

Special Cases

Corollary 4.1

If $\alpha = \beta$ then theorem 4.1 gives

$$\left|\int_a^b p(x,t)\ddot{f}\ (t)dt\right| \le \frac{\|\ddot{f}\|_\infty}{4}\left[\left\{(x-a)\left(a-\left(\frac{a+x}{2}\right)^2\right)+\frac{(x-a)^2}{3}\right\} - \left\{(b-x)\left(b-\left(\frac{b+x}{2}\right)^2\right)-\frac{(b-x)^2}{3}\right\}\right].$$

Proof.

The result follows immediately from theorem 4.1 by taking $\alpha = \beta$.

Corollary 4.2

if $x = \frac{a+b}{2}$, then theorem 4.1 becomes

$$\left|\int_a^b p\left(\frac{a+b}{2},t\right)\ddot{f}\ (t)dt\right| \le \frac{\|\ddot{f}\|_\infty}{2(\alpha+\beta)}\left[\alpha A-\beta B\right.$$

Where

$$A = \left(\frac{b-a}{2}\right)\left[a-\left(\frac{3a+b}{4}\right)^2\right]+\frac{(b-a)^2}{12},$$

And

$$B=\left(\frac{b-a}{2}\right)\left[b-\left(\frac{a+3b}{4}\right)^2\right]-\frac{(b-a)^2}{12}.$$

Proof

The result follows from theorem 4.1 by taking $x=\frac{a+b}{2}$.

Applications

In this section, we discuss some possible applications of the obtained Ostrowski-type inequality. The result established in Theorem 4.1 can be used in the estimation of integral means, numerical integration, and error analysis. Since the inequality provides a bound in terms of the second derivative of the function, it is useful for estimating the deviation between a function and its integral mean over subintervals.

The special cases obtained in Corollaries 4.1 and 4.2 show that the main result can be reduced to symmetric and midpoint-type inequalities by choosing suitable values of the parameters. These forms are particularly useful in midpoint and trapezoidal approximation methods.

The obtained results may also be applied in approximation theory, quadrature formulas, and the study of integral operators. Hence, the developed Montgomery-type identity provides a flexible framework for deriving further inequalities of Ostrowski type.

Conclusion

In this paper, we established a new Montgomery-type identity involving integral means over end intervals by constructing a generalized Peano kernel. Based on this identity, a new Ostrowski-type inequality was derived under the assumption that the second derivative of the function is bounded. The obtained inequality provides an explicit upper bound for the deviation involving integral means over end intervals.

Furthermore, two important special cases were presented, illustrating the flexibility and applicability of the proposed result. The developed inequality extends and generalizes several existing Ostrowski-type inequalities available in the literature. The obtained results may find useful applications in numerical integration, approximation theory, and the analysis of quadrature formulas.

Future work may focus on extending the proposed approach to fractional calculus, convex functions, quantum calculus, and other generalized integral operators.

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