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STUDY OF TWO-DIMENSIONAL QUANTUM BLACK–SCHOLES EQUATION VIA THE NATURAL TRANSFORM DECOMPOSITION METHOD (NTDM)

***1Shahid Khan, 2Imran Khan, 3Dr. Kamran Zakaria, 4Talmeez ur Rehman, 5Aizaz Hussain**

[1,2,5]MS Scholar, Dept. of Mathematics, NED University of Engineering and Technology, Karachi, Pakistan.
[3]PhD Professor (Assistant) at NED University of Engineering and Technology
[4]Institute of Business Administration, Karachi
*[*1]msshahidkhan.rind@gmail.com, [2]imranrajput12ms@gmail.com, [3]zakariakamran@gmail.com,*
[4]turehman@iba.edu.pk, [5]Aizazhussain@gmail.com

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Corresponding Author: *

Abstract

This study presents an analytical–numerical framework for solving the two-dimensional fractional Quantum Black–Scholes equation using the Natural Transform Decomposition Method (NTDM). The conventional Black–Scholes model, when transformed into its Schrödinger-type equivalent, captures the stochastic dynamics of financial markets in a quantum-mechanical sense. To incorporate market irregularities and memory effects, fractional-order temporal derivatives are introduced. The Natural Transform, an integral operator that unifies Laplace and Sumudu transforms, is combined with the Adomian Decomposition Method to form NTDM, enabling fast-convergent series solutions. Analytical and numerical results for the European-style two-asset put option demonstrate the method's efficiency and stability. MATLAB simulations verify the real and imaginary components of the solution and corresponding probability-density distributions. The proposed NTDM framework provides a generalizable tool for solving fractional partial differential equations in quantum finance and related applied-mathematics domains.

OBJECTIVE: This study aims to get the probability of future market stocks in the form of random variables and to estimate the solution of the Schrodinger fractional modified partial differential equation in complex state form for two stocks. A random variable is what is utilized to determine a security's price. The cost and price of the option is equivalent to the solution, and the state of the Schrodinger equation which is a complex state function, according to a probabilistic interpretation.

1. INTRODUCTION

The most simple and common definition and introduction of an option is an agreement among two parties—the option buyer and the option seller—under which the option seller secures the option buyer's right (but not duty) to carry out a certain action (or exercise the option) at a specific future date. The predetermined and pre-calculated price is termed to as the striking price, while the future date is known as the expiry date (2005) [1]. A call option offers its holder the right and opportunity to buy and purchase the underlying asset at a striking price at some time in the future. A put option to its holder at some point in the future provides the right and power to sell the underlying asset at a strike price. According to Peng (2022), there is a wide variety of alternatives which mainly depends upon the decision time. However, European Options are only exercisable on the Expiration Date. Bhatnagar (2022) states that American style alternatives are comparatively more flexible because they may be used whenever you choose, up to the expiration date. They are hence frequently at least as costly as similar European versions.

According to Leong (2022), an alternative is one of the most significant and well-known financial derivatives in the financial market. There are many different sorts of mathematical versions for alternative pricing. One of the most essential ideas in contemporary economic theory is the Black-Scholes model. It is still frequently used today and was created in 1973 by Fischer Black, Robert Merton, and Myron Scholes. It is considered one of the enjoyable ways to ascertain the precise cost of options. Five input variables are needed for the Black-Scholes model: the volatility, the risk-free rate, the term until expiration, the current stock price, and the option's striking price. This form was the first to be widely used for alternative pricing and is sometimes referred to as Black-Scholes-Merton (BSM). To calculate the possible price of options (1973) [2], it takes into account the current stock prices, anticipated dividends, alternative's strike price, anticipated interest rates, a time before expiry, and forecast volatility.

Since this equation can be swiftly solved and transformed into answers, it will become well-known in many areas of study, such as, physics, economics and financial mathematics. Patel (2022) states that the Black-Scholes equation is considered as one of the famous alternative pricing model that is founded on strict presumptions. Even if there is arbitrage in actual financial markets, this equation's underlying premise disallows it. Thus, the Classical Black-Scholes equation was expanded to account for arbitrage opportunities through the work of Contreras et al. [3].

However, Zhao (2022) states that the Black-Scholes equation with arbitrage could be understood through the use and application of the quantum mechanical perspective factor by using the Schrodinger equation in the sense of an imaginary time for an unbound particle. As a result, Contreras developed the Black-Scholes equation, which incorporates the possibility of arbitrage. (2010) [4]

According to Zhang (2022), ,Runge-Kutta calculations of the fourth order are used to model the forecast of the rate at which the US dollar (USD) will exchange for the Indonesian rupiah (IDR) through the use of the nonlinear Schrödinger equation (NLSE). It is possible to compare the NLSE parameters to financial variables in the hopes that they would influence the change in control. Interest rates, hobby fees, return fees, and the gross domestic product is some of these monetary considerations. The NLSE version executes the (IDR/USD) change cost. The NLSE version estimated the expected impacts of the (IDR/USD) change charge areas in comparison with the real facts from the (IDR/USD) change charge, with an error percentage of less than Neath 2.5% each month. This was performed through the use and application of the fourth-order Runge-Kutta numerical technique. Someone based their whole prediction on the Mean Absolute Percentage Error (MAPE) cost estimate, which is 0.48%. However, Kazmi (2022) states that the MAPE cost shows that the less the MAPE cost, the more favorable the predicted effects of the charge modification may be to the actual data from the real exchange rate. [5].

In this paper, a generalized fractional Schrödinger equation is created with fractional spatial and temporal derivatives. To solve the issue for a square potential well and free particles, Fourier transforms, integral transforms, and Laplace transforms are used. The answer is then presented in the form of the Mittag-Leffler function. In addition to this, the Green function for free particles is also included in this paper. Finally, we make a comparison and contrast between the generalized fractional Schrödinger equation's scenarios and those of conventional quantum mechanics. [6]

They produce Schrödinger's linear equations' theoretical and numerical solutions. The unique semi-analytical method is used to do this as an alternative to the traditional Adomian Decomposition Method. The Telescoping Decomposition Method is the name of the new method. The findings can show a strong degree of convergence to their precise forms. The improved version retains accuracy while needing less analytical work; it is extremely dependable and useful. Thus, it is highly recommended in the fields of practical research to solve linked linear and nonlinear differential models. Using a cutting-edge semi-analytical technique, this work has found theoretical and accurate solutions to the linear Schrödinger equations. The TDM is an altered special process ADM. One illustration is the fact that the outcomes exactly match the form. In this case, it is shown that the upgraded version is extremely accurate and productive. Since linearization, interference, or control are not required, there is less analytical work to be done even if accuracy is compromised. Thus, a method for solving linear, nonlinear, stiff, and delayed partial differential equations is provided, with applications in related domains. [7]

As per Guo (2022), in an option pricing of financial concerns, for solving a fractional Black-Scholes-Schrodinger equation, the radial basis functions (RBFs) approach is utilized. Ali (2022) states that the RBFs approach is used to discretize a spatial spin-off process. A straightforward quadrature calculation is used for approximation of the time fractional spinoff which is interpreted inside Caputo's sense. This RBF technique has been theoretically shown using several numerical issues, including the time linear arbitrage bubble scenario and the time step arbitrage bubble case. When fractional order was near 1, the numerical findings were then contrasted with the semi-classical solution. The chosen costs from the RBFs approach are therefore shown in each numerical example to fulfill the semi-classical answer. In this research study, the fractional Black-Scholes-Schrodinger issue is solved through the use of RBFs approach for alternative pricing. By using this technique, a spatial spinoff is discretized, and the approximation of temporal fractional spin-off within Caputo's senses is understood using a straightforward quadrature calculation. The numerical answer and the semi-classical response are then contrasted. (2020)[8].

Using the well-known Black-Scholes differential equation, they determined the cost of a financial derivative. The typical "bit", a system with two potential states could be utilized to describe the uncertain environment of such option pricing. This essay makes the case for the adoption of a brand-new qubit-style of uncertainty. We determine an information-based option price and contrast it with conventional option pricing. [9]

In this work, the non-linear Schrodinger equation served as the foundation for the stock price prediction model. Because this problem lacks an analytical solution, the stock prices were estimated numerically through the use of the fourth-order Runge-Kutta method. In addition, a fourth-order Runge-Kutta method for numerical solutions will be developed. Based on the economic theory and hypothesis, a parameter similarity between the economic factors and nonlinear Schrödinger equation that impact stock prices is found. Some of the anticipated key drivers of stock prices are strike price, volatility , adaptive market potential, average stock return speed, and current stock price movement or growth. Two different companies' stock values were correctly forecasted using this strategy. In comparison to Astra Agro Lestari Tbk, Polychem Indonesia Tbk. (ADMG) has a mean absolute percentage error (MAPE) of 3.48678 percent, according to the prediction findings, as opposed to 0.4633 percent for that company. These MAPE outcomes and results show that the real stock price accord rather well. [10]

The Laplace and Sumudu Transforms are replaced with a new integral transform. It simply converges to each transform through variable conversion. For current Laplace and Sumudu transformations, a table is provided. [11] (2008)

Integral transformations have been discussed in several works and used to address specific boundary value issues. The natural transform is strongly related to the Laplace and Sumudu transforms, as evidenced by reference [12].

According to Ali (2022), the fractional-order heat and wave equations can be solved with the help of the natural-transform decomposition technique. Alghassi (2022) states that the series-shape solutions to the fractional-order heat and wave equations are found with the help of the provided approach. Numerous numerical instances and examples are given so that the system of the natural transform decomposition approach may be understood. The natural transform decomposition method system has demonstrated that many nonlinear problems may be solved with little effort and a high rate of convergence. As a result, the herbal remodels decomposition method is regarded as one of the effective analytical tools, offering a useful solution to fractional-order linear and nonlinear problems, partial differential equations, particularly the heat and wave equation of fractional order. [13]

As per Albosaily (2022), the Schrodinger Equation may be the foundation of quantum mechanics. Quantitative finance might depend heavily on the Black-Sholes equation. Schrodinger's equation for hypothetical time is similar to the Black-Sholes equation. In the Black-Sholes equation, the random variable is employed to find the particle. A random variable is utilised to determine how much security is worth. The Schrodinger equation is used to generate the BS equation. Given that the solution to the Schrodinger equation is a complex state function, the probability of the choice is comparable to the state, hence this interpretation is probabilistic. Quantities, evaluations, and property prices became more accurate in new and effective conditions.

The objective of this study is to calculate the numerical and analytical solutions of the two-dimensional quantum Black-Scholes differential equation model for put and call options with properties. This study will provide an easy-to-understand method for solving the Quantum Black Scholes model's mathematical equations that integrates a portfolio choice made using the European Put option. In addition, we must investigate the numerical final result using the HPM, LHPM, and Wavelet Transform. These methods are applied to the two-dimensional black Scholes-Schrodinger problem to calculate analytical and numerical solutions.

2. METHODOLOGY

- ***Preliminary***

A formal definition of natural transform;

The natural transformation of a function *f*(*t*), defined for all real numbers *t* ≥ *0*, is the function *R*(*u*, *s*), defined by:

$$R(u,s) = \boldsymbol{N}\{F(t)\} = \int_0^\infty (f(ut)e^{-st}\text{dt}. \rightarrow(1)$$

showed that the above integral converges to Laplace transform when *u* = 1, and into Sumudu transform for *s* = 1.

From the Fourier integral, use Natural Transform. The following integral equations are true in the domain for the function f(t), which is piecewise and continuously differentiable in every finite interval and is integrable on the whole real line. $-\infty < t < \infty$.

$$\boldsymbol{F}\{f(t)\} = f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikt}f(t)\,dt\,. \rightarrow (2)$$

$$\boldsymbol{F}^{-\mathbf{1}}\{\text{f(k)}\}\text{= f(t) =} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikt}\, f(k)\text{ dk.} \rightarrow (3)$$

The Fourier and Inverse Fourier transforms, respectively, are represented by the operations F{f(t)} and F^{-1} {f(k)}. And the result of multiplying F{f(t)} and F^{-1} {f(t)} is defined by f(x) gives ,

$$\text{F(x)=}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt}\,dk\ \int_{-\infty}^{\infty} e^{-ikt}\, f(t)dt.\ (4)$$

The Fourier integral formula .

A formal definition of Natural Transform;the of a function *f*(*t*), defined for all real numbers *t* ≥ 0, is the function *R*(*u*, *s*),

$$\boldsymbol{N}^{+}\{f(t)\} = R(s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}\, f(t)dt.\ (5)$$

$$\boldsymbol{N}^{-\mathbf{1}}\{R(s,u)\}\text{ = f(t) =} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}}\ R(s,u)ds.\ (6)$$

Both equations (5 and 6) are linear operators and represent the Natural Transformation of the function f(t) and the inverse natural transform R(s, u), respectively.

Analytical properties :

.Linearity property:

Theorem: if a and b are any constants and $f\ is\ a\ function$ then,

$$N\{af(t)+bg(t)\}=aN\{f(t)\}+bN\{f(t)\} \tag{7}$$

Proof:

$$N\{f(t)\}=\int_0^{\infty} f(ut)e^{-st}dt \quad N\{g(t)\}$$
$$=\int_0^{\infty} g(ut)e^{-st}dt$$

Then a and b are any constants.

$$N\{af(t)+bg(t)\}$$
$$=\int_0^{\infty} e^{-st}\{af(ut)$$
$$+bg(ut)\}dt$$

$$= a\int_0^{\infty} f(ut)e^{-st}dt$$
$$+b\int_0^{\infty} g(ut)e^{-st}dt$$
$$= aN\{f(t)\}+bN\{f(t)\}$$

- **First Translation or Shift property:**

Theorem: Let the function f(t) be a continuous function $t \geq 0$ than,

$$N\left[e^{at}f(t)\right] = \frac{s}{s-u}R\left[\frac{us}{s-u}\right] \tag{8}$$

Proof: The N-transform of $e^{at}f(t)$ is given by

$$N\left[e^{at}f(t)\right]$$
$$=\int_0^{\infty} e^{-(s-au)t}f(ut)dt$$

Therefore by changing variables

$$\left[w=\frac{s-au}{s}t\right]$$

$we\ get$

$$N\left[e^{at}f(t)\right]$$
$$= \frac{s}{s-au}\int_0^{\infty} e^{-sw}f\left(\frac{usw}{s-au}\right)dw$$
$$= \frac{s}{s-u}R\left[\frac{us}{s-u}\right]$$

- ***Scale property :***

Theorem: Property of Change of Scale Consider the function F(at) in set A, where a is a non-zero constant then

$$\boldsymbol{N}^{+}\{\mathrm{f(at)}\}=\frac{1}{a}R[\frac{s}{a},u]. \tag{9}$$

Proof: Using the definition of eq:(5).

$$\boldsymbol{N}^{+}[\ \mathrm{f(at)}\ ]=\frac{1}{u}\int_0^{\infty} e^{\frac{-st}{u}}f(at)\ \mathrm{dt}.$$

Let w= at → t $=\frac{w}{a}$ then dt $=\frac{dw}{a}$.

$$=\frac{1}{u}\int_0^{\infty} e^{\frac{-sw}{au}}\ f(w)\frac{dw}{a}$$

$$=\gg \frac{1}{au}\int_0^{\infty} e^{\frac{-st}{au}}\ \mathrm{f(t)dt}=\frac{1}{a}\mathrm{R}[\frac{s}{a},\mathrm{u}].$$

- ***Natural transform Derivative :***

Theorem: Natural derivative transformation. The natural transform of f n (t), if it is the nth derivative of the function f(t) concerning t, is given by.

$$\boldsymbol{N^{+}[f^{n}(t)]=R_{n}(s,u)}$$
$$\boldsymbol{=\frac{S^{n}}{U^{n}}R(s,u)}$$
$$\boldsymbol{-\sum_{k=0}^{n-1}\frac{S^{n-(k+1)}}{u^{n-k}}f^{k}(0)} \tag{10}$$

Proof: Equation (1)'s first and second derivatives of f(t) are given by their corresponding natural transforms for n=1 and 2, respectively.

$$\boldsymbol{N}^{+}\left[f^{'}(t)\right]=\ R_{1}\left(s,u\right)=\frac{s}{u}R(s,u)-\frac{f(0)}{u}\rightarrow(11)$$

$$\boldsymbol{N}^{+}\left[f^{''}(t)\right]=\ R_{2(s,u)}=\frac{S^{2}R(s,u)-Sf(0)}{U^{2}}-\ \frac{f^{'}(0)^{'}}{u}\rightarrow(12)$$

Assuming that equation (11) is valid for n and demonstrating it for n+1 using equation (12),

$$\boldsymbol{N}^{+}[f^{n+1}(t)]=\boldsymbol{N}^{+}[(f^{n}(t))^{'}]=R_{n+1}(s,u)=$$
$$\frac{s}{u}R_{n}(s,u)-\frac{1}{u}f^{n}(0). \quad (13)$$

$$\frac{s}{u}\left[\frac{s^n}{u^n}R(s,u)-\sum_{k=o}^{n-1}\frac{S^{n-(k+1)}}{u^{n-k}}f^k(0)\right]-\frac{1}{u}f^n(0).$$

$$=\frac{S^{n+1}}{u^{n+1}}R(s,u)-\sum_{k=0}^{n}\frac{S^{n-k}}{u^{(n-k)+1}}f^k(0). \quad (14)$$

- ***Natural Transform Integration:***

Taking into account the integration of function f(t) in set A concerning time t′ in the range (0,t) as w(t) and subsequent integrals as w 2(t) up to w n(t), that is,

$$\text{W(t)} = \int_0^t f(t)dt, W_2(t) = \iint_0^t f(t)(dt)^2 \ldots\ldots . w^n(t) = \int_0^t . \ldots\ldots \int_o^t f(t)(dt)^n. \quad (15)$$

Theorem: Natural integral transformation. The Natural transform of w n(t) is if w n(t) is provided by (4).

$$\boldsymbol{N}^+[W^n(t)] = \frac{u^n}{s^n}R(s,u)$$

$$\text{u}= \int_0^t \ldots\ldots \int_0^t f(t)\,(dt)^n\,, u^n = f(t)(dt); v.dv = e^{\frac{-St}{u}}, v_n = (-)^n\frac{u^n}{s}\quad\frac{-st}{u}.$$

$$=[\ (-)^n\frac{u^n}{s^n}e^{\frac{-st}{u}}W^n(t)]_o^\infty + \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}\frac{u^n}{s^n}\,f(t)dt. \quad (16)$$

- ***Inverse Natural transform:***

The Fourier integral equation was already used to generate the inverse Natural transform (6). We present the formal definition here using the Cauchy residue theorem. We first offer the inverse transforms of the Laplace and Sumudu transforms in the corresponding following theorems.

Theorem: Inverse complex Laplace transform,

$$L^{-1}[\,F(S)] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,F(s)\,ds\,, t > 0. \quad (17)$$

The integral is taken along s = on the complex plane where s = x+iy. F(s) is the Laplace transform. For s = to be on the right of all singularities, the real number is selected.

Theorem: Complex inverse Sumudu transform.

1. $\frac{1}{S}G\left(\frac{1}{S}\right)$ is a monomorphic function with singularities having Re(s) =γ and

2. there exists circular Γ with radius R and positive constants M and K with

$$|\frac{G\left(\frac{1}{S}\right)}{S}| < MR^{-K}$$

Then inverse Sumudu transform is given,

$$S^{-1}[\,G(u)] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G\left(\frac{1}{s}\right)\frac{ds}{s}. \quad (18)$$

3. *Problem Formulation*

For investigating the fair pricing of alternatives in the real financial market, the Black-Scholes-Schrodinger equation is a quantum financial version. The Schrodinger equation is used to interpret the Black-Scholes equation with arbitrage opportunities from the perspective of quantum mechanics. The Black-Scholes-Schrodinger Model is created from the Black-Scholes equation with two assets.

The Black and Scholes is a second-order financial differential equation created using mathematical finance:

V=call option price

S= stock price

r= Risk-free rate

$\sigma = standard\ derivation$ or volatility of stocks return.

T= time before expiring date.

$$\frac{\partial^\alpha p}{\partial t^\alpha}+\frac{1}{2}\sigma_1^2 s_1^2\frac{\partial^2 p}{\partial s_1^2}+\frac{1}{2}\sigma_2^2 s_2^2\frac{\partial^2 p}{\partial s_2^2}+\rho\sigma_1\sigma_2 s_1 s_2\frac{\partial^2 p}{\partial s_1\partial s_2}+ rs_1\frac{\partial p}{\partial s_1}+rs_2\frac{\partial p}{\partial s_2}-rp=0 \rightarrow(19)$$

This is two Dimension Time Fractional ordered European style put option pricing model for two stocks.

By using log stock price;

Let,

$$x = lns_1-\left(r-\frac{1}{2}\sigma_1^2\right)t => \frac{\partial x}{\partial s_1}=\frac{1}{s_1}$$

$$y = lns_2+\left(r-\frac{1}{2}\sigma_2^2\right)t => \frac{\partial y}{\partial s_2}=\frac{1}{s_2}$$

We have $\frac{\partial x}{\partial s_1}=\frac{1}{s_1}$ *and* $\frac{\partial y}{\partial s_2}=\frac{1}{s_2}$;

Now using the chain rule ;

$$\frac{\partial p}{\partial s_1}=\frac{\partial p}{\partial x}\frac{\partial x}{\partial s_1} \quad and \quad \frac{\partial p}{\partial s_2}=\frac{\partial p}{\partial y}\frac{\partial y}{\partial s_2}$$

$$\frac{\partial p}{\partial s_1}=\frac{1}{s_1}\frac{\partial p}{\partial x}\rightarrow(20) \quad and \quad \frac{\partial p}{\partial s_2}=\frac{1}{s_2}\frac{\partial p}{\partial y} \rightarrow(21)$$

$$\frac{\partial^2 p}{\partial s_1^2}=\frac{-1}{s_1^2}\frac{\partial p}{\partial x}+\frac{1}{s_1}\frac{\partial}{\partial s_1}\frac{\partial p}{\partial x} \quad and \quad \frac{\partial^2 p}{\partial s_2^2}=\frac{-1}{s_2^2}\frac{\partial p}{\partial y}+\frac{1}{s_2}\frac{\partial}{\partial s_2}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 p}{\partial s_1^2}=\frac{-1}{s_1^2}\frac{\partial p}{\partial x}+\frac{1}{s_1}\frac{\partial}{\partial x}\frac{\partial x}{\partial s_1}\frac{\partial p}{\partial x} \quad ; \quad \frac{\partial^2 p}{\partial s_2^2}=\frac{-1}{s_2^2}\frac{\partial p}{\partial y}+\frac{1}{s_2}\frac{\partial}{\partial y}\frac{\partial y}{\partial s_2}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 p}{\partial s_1^2}=\frac{1}{s_1^2}\left\{\frac{\partial^2 p}{\partial x^2}-\frac{\partial p}{\partial x}\right\}\rightarrow(24) \quad ;$$

$$\frac{\partial^2 p}{\partial s_2^2}=\frac{1}{s_2^2}\left\{\frac{\partial^2 p}{\partial y^2}-\frac{\partial p}{\partial y}\right\}\rightarrow(25)$$

Differentiating eq.(7) $w.r.t\ s_2$.

$$\frac{\partial^2 p}{\partial s_1\partial s_2}=\frac{1}{s_1}\frac{\partial}{\partial s_2}\frac{\partial p}{\partial x}=\gg\frac{1}{s_1}\frac{\partial}{\partial y}\frac{\partial y}{\partial s_2}\frac{\partial p}{\partial x}\quad=\gg\frac{1}{s_1 s_2}\frac{\partial p}{\partial x\partial y}\text{-----}(26)$$

Putting values from above equations in equation 19 .

$$\frac{\partial^\alpha p}{\partial t^\alpha}+\frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2}+\left(r-\frac{\sigma_1^2}{2}\right)\frac{\partial p}{\partial x}+\left(r-\frac{\sigma_2^2}{2}\right)\frac{\partial p}{\partial y}+\rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y}-rp=0\text{----}(27)$$

be the modified European style put option pricing time-fractional order of Black Scholes model for two assets.

$$x = lns_1-\left(r-\frac{1}{2}\sigma_1^2\right)t$$

$$y = lns_2+\left(r-\frac{1}{2}\sqrt{2}^{2}\right)t$$

$$\frac{\partial p}{\partial t}=\frac{\partial p}{\partial x}\frac{\partial x}{\partial t}$$

$$\boxed{\frac{\partial p}{\partial t}=\frac{-\partial p}{\partial x}\left(r-\frac{1}{2}\sigma_1^2\right)}$$

$$\&\frac{-\partial p}{\partial t}=\frac{\partial p}{\partial u}\left(r-\frac{1}{2}\sigma_1^2\right)$$

$$\frac{\partial p}{\partial t}=\frac{\partial p}{\partial y}\frac{\partial y}{\partial t}$$

$$\boxed{\frac{\partial p}{\partial t}=\frac{\partial p}{\partial y}\left(r-\frac{1}{2}\sigma_1^2\right)}\text{----------}(28)$$

equation (19) becomes.

$$\frac{\partial^{\propto} p}{\partial t^{\propto}} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial t} + \frac{\partial p}{\partial t} + + p\,\sigma_1\sigma_2\frac{\partial^2 p}{\partial x\,\partial y} - rp = 0$$

(29)

To obtain Schrodinger –Black-Scholes model time-fractional order, and European order of put option pricing for stock, we consider ;

$$\text{Consider} t^{\alpha} = iT^{\alpha}; \quad \text{Let X } = T^{\alpha};$$

$$t^{\alpha} = iX;$$

$$\frac{\partial t^{\alpha}}{\partial X} = i \Rightarrow\!\!> \partial t^{\alpha} = i\partial X;$$

$$\partial t^{\alpha} = - i\partial T^{\alpha}$$

$$\frac{\partial^{\alpha} p}{i\partial T^{\alpha}} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y} - rp = 0;$$

$$i\frac{\partial^{\alpha} p}{\partial T^{\alpha}} = \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y} - rp = 0; \qquad (29)$$

Despite this, the Black-Scholes-Schrodinger equation does not describes financial market completely.

Therefore, in the Black-Scholes-Schrodinger equation, fractional calculus is used to characterize the variability in prices and to explain occurrences in the financial market, notably in the field of log-price probability. The fractional coefficients for the Black-Scholes-Schrodinger equation are:

$$i\frac{\partial^{\alpha} p}{\partial T^{\alpha}} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y} - rp = 0; \rightarrow \quad (30)$$

where is a fractional order,0<α≤1. It may also be thought of as a model parameter, in which each model will give its answer. When α=1, equation (30) becomes simpler and becomes the original Black-Scholes-Schrodinger equation.

4. Problem Formulation

For investigating the fair pricing of alternatives in the real financial market, the Black-Scholes-Schrodinger equation is a quantum financial version. The Schrodinger equation is used to interpret the Black-Scholes equation with arbitrage opportunities from the perspective of quantum mechanics. The Black-Scholes-Schrodinger Model is created from the Black-Scholes equation with two assets.

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T= time before expiring date.

$$\frac{\partial^{\alpha} p}{\partial t^{\alpha}} + \frac{1}{2}\sigma_1^2 s_1^2\frac{\partial^2 p}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 s_2^2\frac{\partial^2 p}{\partial s_2^2} + \rho\sigma_1\sigma_2 s_1 s_2\frac{\partial^2 p}{\partial s_1\partial s_2} + rs_1\frac{\partial p}{\partial s_1} + rs_2\frac{\partial p}{\partial s_2} - rp = 0 \rightarrow (31)$$

This is two Dimension Time Fractional ordered European style put option pricing model for two stocks.

By using log stock price;

Let,

$$x = lns_1 - \left(r - \frac{1}{2}\sigma_1^2\right)t => \frac{\partial x}{\partial s_1} = \frac{1}{s_1}$$

$$y = lns_2 + \left(r - \frac{1}{2}\sigma_2^2\right)t => \frac{\partial y}{\partial s_2} = \frac{1}{s_2}$$

We have $\frac{\partial x}{\partial s_1} = \frac{1}{s_1}$ *and* $\frac{\partial y}{\partial s_2} = \frac{1}{s_2}$;

Now using the chain rule ;

$$\frac{\partial p}{\partial s_1} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial s_1} \quad and \quad \frac{\partial p}{\partial s_2} = \frac{\partial p}{\partial y}\frac{\partial y}{\partial s_2}$$

$$\frac{\partial p}{\partial s_1} = \frac{1}{s_1}\frac{\partial p}{\partial x} \rightarrow (31) \quad and \quad \frac{\partial p}{\partial s_2}\frac{1}{s_2}\frac{\partial p}{\partial y} \rightarrow (32)$$

$$\frac{\partial^2 p}{\partial s_1^2} = \frac{-1}{s_1^2}\frac{\partial p}{\partial x} + \frac{1}{s_1}\frac{\partial}{\partial s_1}\frac{\partial p}{\partial x} \quad and \quad \frac{\partial^2 p}{\partial s_2^2} = \frac{-1}{s_2^2}\frac{\partial p}{\partial y} + \frac{1}{s_2}\frac{\partial}{\partial s_2}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 p}{\partial s_1^2} = \frac{-1}{s_1^2}\frac{\partial p}{\partial x} + \frac{1}{s_1}\frac{\partial}{\partial x}\frac{\partial x}{\partial s_1}\frac{\partial p}{\partial x} \quad ; \quad \frac{\partial^2 p}{\partial s_2^2} = \frac{-1}{s_2^2}\frac{\partial p}{\partial y} + \frac{1}{s_2}\frac{\partial}{\partial y}\frac{\partial y}{\partial s_2}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 p}{\partial s_1^2} = \frac{1}{s_1^2}\left\{\frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}\right\} \rightarrow (iii) \quad ; \quad \frac{\partial^2 p}{\partial s_2^2} = \frac{1}{s_2^2}\left\{\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial y}\right\} \rightarrow (33)$$

Differentiating eq. $w.r.t\ s_2$.

$$\frac{\partial^2 p}{\partial s_1 \partial s_2} = \frac{1}{s_1}\frac{\partial}{\partial s_2}\frac{\partial p}{\partial x} \Rightarrow\!\!> \frac{1}{s_1}\frac{\partial}{\partial y}\frac{\partial y}{\partial s_2}\frac{\partial p}{\partial x} \quad \Rightarrow\!\!> \frac{1}{s_1 s_2}\frac{\partial p}{\partial x \partial y} \text{-----}(34)$$

Putting eq.(30)(31),(32),(33),(34) .

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma_1^2\left(\frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}\right) + \frac{1}{2}\sigma_2^2\left(\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial y}\right) + \rho\sigma_1\sigma_2\frac{\partial^2 p}{\partial x \partial y} + r\frac{\partial p}{\partial x} + r\frac{\partial p}{\partial y} - rp = 0;$$

$$\frac{\partial^\alpha p}{\partial t^\alpha} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} + \left(r - \frac{\sigma_1^2}{2}\right)\frac{\partial p}{\partial x} + \left(r - \frac{\sigma_2^2}{2}\right)\frac{\partial p}{\partial y} + \rho\sigma_1\sigma_2\frac{\partial p}{\partial x \partial y} - rp = 0\text{----}(35)$$

be the modified European style put option pricing time-fractional order of Black Scholes model for two assets.

$$x = lns_1 - \left(r - \frac{1}{2}\sigma_1^2\right) t$$

$$y = lns_2 + \left(r - \frac{1}{2}\sqrt{2}^2\right) t$$

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial t}$$

$$\boxed{\frac{\partial p}{\partial t} = \frac{-\partial p}{\partial x}\left(r - \frac{1}{2}\sigma_1^2\right)} \qquad \&$$

$$\frac{-\partial p}{\partial t} = \frac{\partial p}{\partial u}\left(r - \frac{1}{2}\sigma_1^2\right)\text{-------}(36)$$

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial y}\frac{\partial y}{\partial t}$$

$$\boxed{\frac{\partial p}{\partial t} = \frac{\partial p}{\partial y}\left(r - \frac{1}{2}\sigma_1^2\right)} \qquad \text{----------}(37)$$

$\Rightarrow$

$$\frac{\partial^\propto p}{\partial t^\propto} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} - \frac{\partial p}{\partial t} + \frac{\partial p}{\partial t} + + p\,\sigma_1\sigma_2\frac{\partial^2 p}{\partial x\,\partial y} - rp = 0$$

$$\frac{\partial^\alpha p}{\partial t^\alpha} + \frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial p}{\partial x \partial y} - rp = 0\text{----}(38)$$

To obtain Schrodinger –Black-Scholes model time-fractional order, and European order of put option pricing for stock, we consider ;

Consider$t^\alpha = iT^\alpha$; Let $X = T^\alpha$;

$$t^\alpha = iX;$$

$$\frac{\partial t^\alpha}{\partial X} = i \Rightarrow\!\!> \partial t^\alpha = i\partial X;$$

$$\partial t^\alpha = - i\partial T^\alpha$$

$$\frac{\partial^\alpha p}{i\partial T^\alpha}+\frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y}-rp=0;$$

$$i\frac{\partial^\alpha p}{\partial T^\alpha}=\frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y}-rp=0;$$
(39)

Despite this, the Black-Scholes-Schrodinger equation does not describes financial market completely.

Therefore, in the Black-Scholes-Schrodinger equation, fractional calculus is used to characterize the variability in prices and to explain occurrences in the financial market, notably in the field of log-price probability. The fractional coefficients for the Black-Scholes-Schrodinger equation are:

$$i\frac{\partial^\alpha p}{\partial T^\alpha}+\frac{\sigma_1^2}{2}\frac{\partial^2 p}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial p}{\partial x\partial y}-rp=0;\rightarrow\quad(40)$$

where is a fractional order,0<α≤1:: It may also be thought of as a model parameter, in which each model will give its answer. When α=1, equation becomes simpler and becomes the original Black-Scholes-Schrodinger equation.

Table of Symbols and Parameters

Symbol	Description
S₁, S₂	Underlying asset prices 1 and 2
t, T	Time and maturity
R	Risk-free interest rate
σ₁, σ₂	Volatilities of assets 1 and 2
Ρ	Correlation coefficient between the two assets (−1 ≤ ρ ≤ 1)
V(S₁,S₂,t)	Option value (price)
K₁, K₂	Strike prices for assets 1 and 2
Α	Fractional derivative order (0 < α ≤ 1)
I	Imaginary unit (√−1)
ψ(x,y,t)	Quantum wave function corresponding to V
{·}, $^{-1}${·}	Natural Transform and its inverse
Γ(·)	Gamma function
D_t^α	Caputo fractional derivative of order α
Φ(x,y,t)	Auxiliary function in decomposition
P(x,y,t)	Probability density |ψ|²
λ, ħ	Model scaling (if used) and reduced Planck-like constant

5. SOLUTION OF SCHRODINGER FRACTIONAL ORDER PDE BY USING NATURAL TRANSFORM

Natural Transform on Black Scholes Schrodinger equation:

a) $i\frac{\partial^\alpha v}{\partial T^\alpha}+\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial v}{\partial x\partial y}-rv=0$

$$\frac{\partial^\alpha v}{\partial T^\alpha}=-i\left\{-\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}-\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}-\rho\sigma_1\sigma_2\frac{\partial v}{\partial x\partial y}+rv\right\}$$

$$\frac{\partial^\alpha v}{\partial T^\alpha}=i\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial v}{\partial x\partial y}-rv\right\}\text{……(41)}$$

$0<\alpha\leq 2$ with the initial condition;

v(x, y, o) = 0 : v_t(x, y, o) = z^2 ;

Taking the natural transform of Equation (41);

$$\frac{s^{\alpha}}{u^{\alpha}}N^{+}[\,V(x,y,t)] - \frac{s^{\alpha-1}}{u^{\alpha}}\,v(x,y,0) - \frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o) = iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

$$\frac{s^{\alpha}}{u^{\alpha}}N^{+}[\,V(x,y,t)] = \frac{s^{\alpha-1}}{u^{\alpha}}\,v(x,y,0) + \frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o) + iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

$$N^{+}[\,V(x,y,t)] = \frac{u^{\alpha}}{s^{\alpha}}[\{\frac{s^{\alpha-1}}{u^{\alpha}}\,v(x,y,0) + \frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)\} + iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

Applying inverse natural transform;

$$[\,V(x,y,t)] = N^{-}\{\frac{v(x,y,0)}{S} + \frac{UV_t(x,y,o)\}}{S^2}\}\cdot + N^{-}\{\frac{u^{\alpha}}{s^{\alpha}}iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

Using the ADM procedure, we get;

$$v_o(x,y,t) = N^{-}\{\frac{v(x,y,0)}{S} + \frac{UV_t(x,y,o)}{S^2}\}\ ;$$

$$v_o(x,y,t) = tz^2;$$

$$\sum_{j=0}^{\infty} v_{j+1}(\mathrm{x,\ y,\ t}) = -iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\sum_{j=0}^{\infty} v_{xxj} + \frac{\sigma_2^2}{2}\sum_{j=0}^{\infty} v_{yyj} + \rho\sigma_1\sigma_2\sum_{j=0}^{\infty} v_{xyj} - rv_0\right\}.$$

For j = 0;

$$v_1(\mathrm{x,\ y,\ t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_0}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_o}{\partial x\partial y} - rv_o\right\}$$

$$v_1(\mathrm{x,\ y,\ t}) = -irt\cdot z^2\frac{t^{\alpha}}{\Gamma(\alpha+1)};$$

The subsequent terms are;

$$v_2\ (\mathrm{x,\ y,\ t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_1}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_1}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_1}{\partial x\partial y} - rv_1\right\};$$

$$v_2\ (\mathrm{x,\ y,\ t}) = -\,r^2tz^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)};$$

$$v_3\ (\mathrm{x,\ y,\ t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_2}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_2}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_2}{\partial x\partial y} - rv_2\right\};$$

$$v_3\ (\mathrm{x,\ y,\ t}) = ir^3tz^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)};\qquad v_4(\mathrm{x,y,t}) = r^4tz^2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$v_5(\mathrm{x,y,t}) = -\,\mathrm{i}r^5tz^2\frac{t^{5\alpha}}{\Gamma(5\alpha+1)};$$

$$v_6(\mathrm{x,y,t}) = -\,r^6tz^2\frac{t^{6\alpha}}{\Gamma(6\alpha+1)};$$

$$v_7(\mathrm{x,y,t}) = \mathrm{i}r^7tz^2\frac{t^{7\alpha}}{\Gamma(7\alpha+1)};$$

$$v_8(\mathrm{x,y,t}) = r^8tz^2\frac{t^{8\alpha}}{\Gamma(8\alpha+1)};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

……..

The NTDM solution for;

$$\mathrm{V(x,y,t)} = v_o(x,y,t) + v_1(\mathrm{x,y,t}) + v_2(\mathrm{x,y,t}) + v_3(\mathrm{x,y,t}) + v_4(\mathrm{x,y,t}) + v_5(\mathrm{x,y,t}) + v_6(\mathrm{x,y,t}) + v_7(\mathrm{x,y,t}) + v_7(\mathrm{x,y,t})\ldots$$

$$\mathrm{V(x,y,t)} = tz^2 - irt\cdot z^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} - r^2tz^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ir^3tz^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + r^4tz^2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} -$$

$ir^5tz^2\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-r^6tz^2\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+ir^7tz^2\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}+$
$r^8tz^2\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$ ….

…

V(x,y,t)=tz^2{1-
$ir\frac{t^{\alpha}}{\Gamma(\alpha+1)}-r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+ir^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+$
$r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-ir^5te^z\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-r^6te^z\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+$
$ir^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}+\ r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$ …………}

V(x, y ,t) = tz^2
[(1$-r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-r^6\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+$
$r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$…) -i
($r\frac{t^{\alpha}}{\Gamma(\alpha+1)}-r^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+r^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-$
$r^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}$……)]

when r = 1 then the NTDM solution α = 1;

V(x, y ,t)= tz^2[(1-$r^2\frac{t^2}{2!}+r^4\frac{t^4}{4!}-r^6\frac{t^{\,6}}{6!}+$
$r^8\frac{t^8}{8!}$….) $-i(rt-r^3\frac{t^{\,3}}{3!}+r^5\frac{t^{\,5}}{5!}-r^7\frac{t^{\,7}}{7!}$…)

V(x, y ,t)= tz^2 [(1$-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^{\,6}}{6!}+\frac{t^8}{8!}$….) $-i(t-$
$\frac{t^{\,3}}{3!}+\frac{t^{\,5}}{5!}-\frac{t^{\,7}}{7!}$…)]

V(x, y ,t)= $tz^2(Cost-iSint)$

This is the solution.

b) $$\frac{\partial^{\alpha}v}{\partial T^{\alpha}}=i\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}$$

$0<\alpha\leq 2$ with an initial condition;

V(x, y, o) = 0 : V_t(x, y, o) = e^z ;

Taking the natural transform of Equation (1);

$$\frac{s^{\alpha}}{u^{\alpha}}N^{+}[V(x,y,t)]-\frac{s^{\alpha-1}}{u^{\alpha}}v(x,y,0)-\frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)=iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}.$$

$$\frac{s^{\alpha}}{u^{\alpha}}N^{+}[V(x,y,t)]=\frac{s^{\alpha-1}}{u^{\alpha}}v(x,y,0)+\frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)+iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}.$$

$$N^{+}[V(x,y,t)]=\frac{u^{\alpha}}{s^{\alpha}}[\{\frac{s^{\alpha-1}}{u^{\alpha}}v(x,y,0)+\frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)\}+iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}.$$

Applying inverse natural transform;

$$[V(x,y,t)]=N^{-}\{\frac{v(x,y,0)}{S}+\frac{UV_t(x,y,o)\}}{S^2}\}+N^{-}\{\frac{u^{\alpha}}{s^{\alpha}}iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}.$$

Using the ADM procedure, we get;

$$v_o(x,y,t)=N^{-}\{\frac{v(x,y,0)}{S}+\frac{UV_t(x,y,o)}{S^2}\};$$

$$v_o(x,y,t)=te^z;$$

$$\sum_{j=0}^{\infty}v_{j+1}(x,y,t)=-iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\sum_{j=0}^{\infty}v_{xxj}+\frac{\sigma_2^2}{2}\sum_{j=0}^{\infty}v_{yyj}+\rho\sigma_1\sigma_2\sum_{j=0}^{\infty}v_{xyj}-rv_0\right\}.$$

For j = 0;

$$v_1(x,y,t)=iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_0}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v_0}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v_o}{\partial x\partial y}-rv_o\right\}$$

v_1(x , y, t) = $-irt\cdot e^z\frac{t^\alpha}{\Gamma(\alpha+1)}$;

The subsequent terms are;

v_2 (x , y, t) = $iN^-\{\frac{u^\alpha}{s^\alpha}N^+\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_1}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v_1}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v_1}{\partial x\partial y}-rv_1\right\}$;

v_2 (x , y, t) = - $r^2te^z\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$;

v_3 (x, y, t)= $iN^-\{\frac{u^\alpha}{s^\alpha}N^+\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_2}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v_2}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v_2}{\partial x\partial y}-rv_2\right\}$;

v_3 (x , y, t)= $ir^3te^z\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$; $v_4(\mathrm{x},\mathrm{y},\mathrm{t}) = r^4te^z\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$

$v_5(\mathrm{x},\mathrm{y},\mathrm{t}) =-\mathrm{i}r^5te^z\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$;

$v_6(\mathrm{x},\mathrm{y},\mathrm{t}) =- r^6te^z\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}$;

$v_7(\mathrm{x},\mathrm{y},\mathrm{t}) = \mathrm{i}r^7te^z\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}$;

$v_8(\mathrm{x},\mathrm{y},\mathrm{t}) = r^8te^z\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$; ....

....

The NTDM solution for;

V(x ,y, t) = $v_o(x,y,t)+v_1$(x , y, t)+v_2 (x , y, t)+v_3 (x , y, t)+$v_4(\mathrm{x},\mathrm{y},\mathrm{t})+v_5(\mathrm{x},\mathrm{y},\mathrm{t})+v_6(\mathrm{x},\mathrm{y},\mathrm{t})+v_7(\mathrm{x},\mathrm{y},\mathrm{t})+v_7(\mathrm{x},\mathrm{y},\mathrm{t})$…

V(x, y ,t) =

$$te^z-irt\cdot e^z\frac{t^\alpha}{\Gamma(\alpha+1)}-r^2te^z\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+ir^3te^z\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+r^4te^z\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\mathrm{i}r^5te^z\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-r^6te^z\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+\mathrm{i}r^7te^z\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}+r^8te^z\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}\ldots.$$

…

V(x,y,t)=te^z{1-$ir\cdot\frac{t^\alpha}{\Gamma(\alpha+1)}-r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+ir^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\mathrm{i}r^5te^z\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-r^6te^z\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+\mathrm{i}r^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}+r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$............}

V(x, y ,t) = te^z [$(1-r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-r^6\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}\ldots)$ -i $(r\frac{t^\alpha}{\Gamma(\alpha+1)}-r^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+r^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}-r^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}\ldots\ldots)$]

when $r = 1$ *then the NTDM solution* $\alpha = 1$;

V(x,y ,t)= te^z [(1-$r^2\frac{t^2}{2!}+r^4\frac{t^4}{4!}-r^6\frac{t^6}{6!}+r^8\frac{t^8}{8!}\ldots.)-i(rt-r^3\frac{t^3}{3!}+r^5\frac{t^5}{5!}-r^7\frac{t^7}{7!}\ldots)$

V(x, y ,t)= te^z [(1-$\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\frac{t^8}{8!}\ldots.)-i(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\ldots)$]

V(x, y ,t)= $te^z(Cost-iSint)$ (42)

This is the solution.

:) $\frac{\partial^\alpha v}{\partial T^\alpha}=i\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}$

0 <α ≤2 with the initial condition;

v(x, y, o) = 0 : v_t(x, y, o) = e^{it}

Taking the natural transform of Equation (1);

$$\frac{s^\alpha}{u^\alpha}N^+[V(x,y,t)]-\frac{s^{\alpha-1}}{u^\alpha}v(x,y,0)-\frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)=iN^+\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y}-rv\right\}.$$

$$\frac{s^{\alpha}}{u^{\alpha}}N^{+}[\,V(x,y,t)] = \frac{s^{\alpha-1}}{u^{\alpha}}\;v(x,y,0) + \frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o) + iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

$$N^{+}[\,V(x,y,t)] = \frac{u^{\alpha}}{s^{\alpha}}[\{\frac{s^{\alpha-1}}{u^{\alpha}}\;v(x,y,0) + \frac{s^{\alpha-2}}{u^{\alpha-1}}V_t(x,y,o)\} + iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

Applying inverse natural transform;

$$[\,V(x,y,t)] = N^{-}\{\frac{v(x,y,0)}{S} + \frac{UV_t(x,y,o)\}}{S^2}\}\cdot + N^{-}\{\frac{u^{\alpha}}{s^{\alpha}}iN^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial}{\partial x\partial y} - rv\right\}.$$

Using the ADM procedure, we get;

$$v_o(x,y,t) = N^{-}\{\;\frac{v(x,y,0)}{S} + \frac{UV_t(x,y,o)}{S^2}\}\;;$$

$$v_o(x,y,t) = te^{it};$$

$$\sum_{j=0}^{\infty} v_{j+1}(\mathrm{x,\,y,\,t}) = -iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\sum_{j=0}^{\infty} v_{xxj} + \frac{\sigma_2^2}{2}\sum_{j=0}^{\infty} v_{yyj} + \rho\sigma_1\sigma_2\sum_{j=0}^{\infty} v_{xyj} - rv_0\right\}.$$

For j = 0;

$$v_1(\mathrm{x\,,\,y,\,t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_0}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_o}{\partial x\partial y} - rv_o\right\}$$

$$v_1(\mathrm{x\,,\,y,\,t}) = \;-irt{\cdot}e^{it}\frac{t^{\alpha}}{\Gamma(\alpha+1)};$$

The subsequent terms are;

$$v_2\;(\mathrm{x\,,\,y,\,t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_1}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_1}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_1}{\partial x\partial y} - rv_1\right\};$$

$$v_2\;(\mathrm{x\,,\,y,\,t}) = -\,r^2te^{it}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)};$$

$$v_3\;(\mathrm{x,\,y,\,t}) = iN^{-}\{\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left\{\frac{\sigma_1^2}{2}\frac{\partial^2 v_2}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 v_2}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 v_2}{\partial x\partial y} - rv_2\right\};$$

$$v_3\;(\mathrm{x\,,\,y,\,t}) = \;\;ir^3te^{it}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)};\qquad v_4(\mathrm{x\,,y,t}) = r^4te^{it}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$v_5(\mathrm{x\,,y,t}) =-\,\mathrm{i}r^5te^{it}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}\,;$$

$$v_6(\mathrm{x\,,y,t}) =-\,r^6te^{it}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)};$$

$$v_7(\mathrm{x\,,y,t}) = \mathrm{i}r^7te^{it}\frac{t^{7\alpha}}{\Gamma(7\alpha+1)};$$

$$v_8(\mathrm{x\,,y,t}) = r^8te^{it}\frac{t^{8\alpha}}{\Gamma(8\alpha+1)};\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

……..

The NTDM solution for;

$$\mathrm{V(x\,,y,\,t)} = v_o(x,y,t) + v_1(\mathrm{x\,,\,y,\,t}) + v_2\;(\mathrm{x\,,\,y,\,t}) + v_3\;(\mathrm{x\,,\,y,\,t}) + v_4(\mathrm{x\,,y,t}) + v_5(\mathrm{x\,,y,t}) + v_6(\mathrm{x\,,y,t}) + v_7(\mathrm{x\,,y,t}) + v_7(\mathrm{x\,,y,t})\ldots$$

$$\mathrm{V(x,\,y\,,t)} = \;te^{it} - irt{\cdot}e^{it}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - r^2te^{t}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \;-\; ir^3te^{t}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + r^4te^{it}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \mathrm{i}r^5te^{it}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - r^6te^{it}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \mathrm{i}r^7te^{it}\frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \;r^8te^{it}\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}\ldots.$$

$$\mathrm{V(x,y,t)}=te^{it}\{1- ir^{\cdot}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ir^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \mathrm{i}r^5te^{z}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - r^6te^{z}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \mathrm{i}r^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \;r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}\;\ldots\ldots\ldots\ldots\}$$

$$\mathrm{V(x,\,y\,,t)} = te^{it}$$

$$[\,(1-r^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + r^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - r^6\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + r^8\frac{t^{8\alpha}}{\Gamma(8\alpha+1)}\ldots)\;\mathrm{-i}$$

$$\left(r\frac{t^{\alpha}}{\Gamma(\alpha+1)} - r^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + r^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - r^7\frac{t^{7\alpha}}{\Gamma(7\alpha+1)} \ldots\ldots \right) \Big]$$

$when\ r = 1\ then\ the\ NTDM\ solution\ \alpha = 1;$

$$V(x,y,t) = te^{it}\Big[\left(1 - r^2\frac{t^2}{2!} + r^4\frac{t^4}{4!} - r^6\frac{t^6}{6!} + r^8\frac{t^8}{8!} \ldots\right) - i\left(rt - r^3\frac{t^3}{3!} + r^5\frac{t^5}{5!} - r^7\frac{t^7}{7!} \ldots\right)$$

$$V(x,y,t) = te^{it}\Big[\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \ldots\right) - i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \ldots\right) \Big]$$

$$V(x,y,t) = te^{it}(Cost - iSint)$$

$$V(x,y,t) = t(Cost + iSint)(Cost - iSint);$$

$$V(x,y,t) = t(cos^2(t) + sin^2(t))$$

This is the solution.

EXAMPLE: Choice type: Put option: Take into account the information below.S_1 = Price of stock 1

S_2 = Price of stock 2

S_1	20	40	70	100	150
S_2	50	80	120	180	200

Initial Condition :

P(S_1, S_2,t) =Max ((90-sin(s_1+2s_2)i , 0)

- Stock exercise price 1= $30
- Stock exercise price 2 = $90
- Maximum Stock exercise price = $90

  Strategy: Put option.
- Exercise data or the month of expiration information = 2 months
- $\propto = 0.25$
- S.D of stock 1 = σ_1 = 28%
- S.D of stock 2 = σ_2 = 25 %
- Proportion of stock 1 = w_1 = 2 & proportion of stock 2=w_2 =5
- Risk-free rate of return = 4%
- Correlation coefficient between = 65% Stock 1 and stock 2

  The real solution is:

P =-1.9249cos(s_1 + 2.0s_2)- 1.0495sin(s_1 + 2.0s_2)+ 89.934

Imaginary Solution is:

P= 0.55521cos(s_1 + 2.0s_2) - 2.0238sin(s_2 + 2.0s_2) - 2.5361;

Probability density functions for wave Put option price function:

$$P(s_1, s_2) = (-1.9249cos(s_1 + 2.0s_2) - 1.0495sin(s_1 + 2.0s_2) + 89.934)^2 + (0.55521cos(s_1 + 2.0s_2) - 2.0238sin(s_2 + 2.0s_2) - 2.5361)^2$$

Normalized Probability density functions for wave put option price function

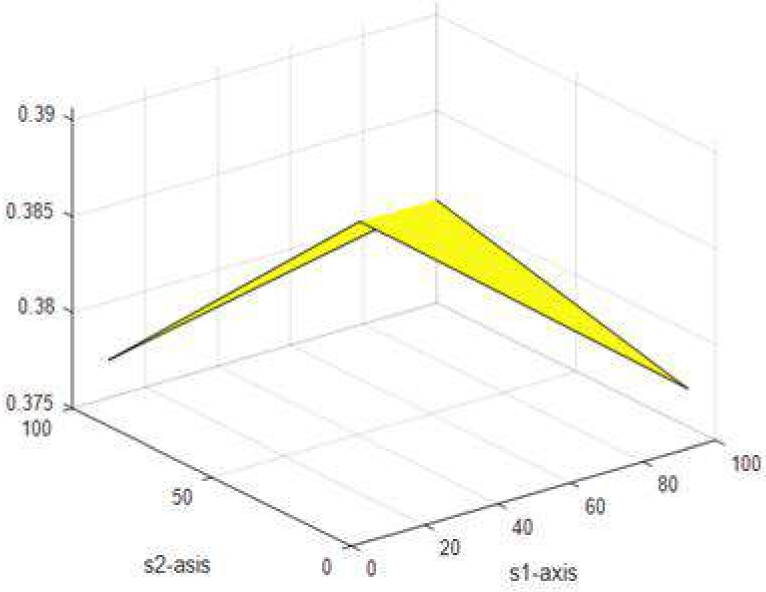
$$P(s_1, s_2) = 0.0068979\,[\,(-1.9249cos(s_1 + 2.0s_2) - 1.0495sin(s_1 + 2.0s_2) + 89.934)^2 + (\,0.55521cos(s_1 + 2.0s_2) - 2.0238sin(s_2 + 2.0s_2) - 2.5361)^2]$$

Probability of Put option for stocks-price at least $70 and $120

P = 0.6080 = 60.80%

Probability of Put option for stocks-price at most $100 and $180

P = 0.7352 = 73.52%



6. DISCUSSION

The Black Scholes Model may be understood in the context of quantum physics by looking at the Schrodinger Wave Equation of a free particle. The Schrodinger Black Scholes model illustrates the division between the financial and economic markets. The Schrodinger Black Sholes Model is credited with establishing "Quantum Finance," which may be used to evaluate call and put option pricing premiums at the exercise price for hypothetical time t.

Quantitative and financial markets are built on the Black Scholes model, but quantum science is based on the Schrodinger equation. To better understand and evaluate the parallels between the two well-known models, the essential connections between the Black Scholes Model in finance and the Schrodinger Equation in quantum physics are given here.

- In Quantum mechanics, the position of a free particle is a random variable.
- In finance, the price of share stock is a random variable.

- In Quantum Mechanics, Schrodinger Equation estimates the complex state of function.
- In Finance, BS Model computes the call/put option price stock at the exercise price.
- In Quantum Mechanics, Schrodinger Equation is the complex P.D.E that estimates both real and imaginary solutions.

Only a few papers have been written about the issue of applying additional well-known aspects of quantum physics to finance, particularly when using an extra general model of the Schrodinger differential equation. This model essentially aims to achieve a Black- Scholes-Schrodinger model by addressing data and uncertainty issues. Our work on the two-dimensional Schrodinger Black-Scholes equation, which can be converted into a Quantum Black-Scholes equation, differs from that of many other researchers who focus on the same Quantum Black-Scholes equation.

In the company's corporate world, finance is one of the sectors that is growing the fastest. Prior to today, business experts and business students ran the global corporation Finance; now, mathematicians and computer scientists are in charge. The manipulation of the danger posed by financial markets has evolved in many ways. In pricing economic derivatives, numbers, in particular when there may be no closed-form analytical solution, are crucial.

7. CONCLUSION

In this work, a unified analytical framework based on the Natural Transform has been developed for solving the fractional Schrödinger equation and the fractional Black–Scholes equation under Caputo fractional derivatives. The proposed approach provides closed-form and semi-analytical solutions without relying on auxiliary polynomial expansions or iterative correction schemes commonly employed in decomposition-based and perturbation methods.

The results demonstrate that the Natural Transform offers a computationally efficient and analytically transparent alternative to classical Laplace- and Fourier-based techniques. Limiting-case analysis confirms the consistency of the obtained solutions with classical integer-order models, while comparison with existing methods highlights the reduced algebraic complexity and improved solution structure.

The key contribution of this study lies in the methodological novelty of applying the Natural Transform in a unified manner to fractional-order models arising in both quantum mechanics and financial mathematics. This dual applicability underscores the versatility of the proposed framework and its potential extension to a broader class of fractional partial differential equations.

Future research may focus on extending the present approach to nonlinear fractional models, variable-order derivatives, and multi-dimensional systems, as well as exploring numerical-analytical hybrid formulations for real-world applications.

The results confirm that the Natural Transform serves as a symmetry-preserving analytical tool for fractional differential equations, offering an efficient and structurally consistent solution framework.

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