

DEVELOPMENT OF ACCELERATED THIRD ORDER RUNGE KUTTA METHOD FOR NON-AUTONOMOUS ORDINARY DIFFERENTIAL EQUATION

Muhammad Daud Kandhro¹, Raja Faisal Soomro², Muhammad Imran Soomro³,
Ayaz Ali Mallah⁴

¹Institute of Mathematics and Computer Science, University of Sindh, Jamshoro

^{1,2,3}School Education and Literacy Department, Govt. of Sindh

⁴Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro

¹daud.kandhro@gmail.com, ²enr.rajafaisalsoomro@gmail.com, ³soomro.emran@gmail.com,

⁴ayaz.mallah@faculty.muett.edu.pk

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Corresponding Author: *

Muhammad Daud
Kandhro

Abstract

In this article, we derive well-organized third order numerical technique for IVP of ODE's including partial derivative which has enhanced its competency regarding truncation error. This new accelerated proposed scheme is used to evaluate numerically ODE's with initial condition. For this, several numerical examples are tested. Last Absolute Error, Maximum Absolute Error and CPU Time has calculated. Convergence of an accelerated proposed method is diagnosed by testing several problems and its consequence verified with available methods. Numerical results are given for proposed scheme to illustrate the accuracy and efficiency. The comparisons of the proposed and existing schemes having same order of local accuracy are discussed. Based on the results, the proposed scheme gives the better result than few same order methods. MATLAB 2023a is preferred for numerically and graphically analysis of methods

1. INTRODUCTION

Numerical analysis plays a fundamental role in solving mathematical problems that do not admit exact analytical solutions, particularly in the context of non-autonomous ordinary differential equations (ODEs). It focuses on the development and analysis of algorithms that provide approximate solutions with high accuracy and computational efficiency. In this study, numerical analysis is employed to construct an accelerated third-order Runge-Kutta method, ensuring that the proposed scheme satisfies consistency, stability, and convergence criteria. The performance of the method is evaluated by

comparing it with classical schemes, emphasizing reduced computational cost and improved accuracy for time-dependent systems.

The numerical solution of ordinary differential equations involves approximating the solution of initial value problems using discrete methods. For non-autonomous ODEs of the form $y' = f(x, y)$, analytical solutions are often difficult or impossible to obtain, making numerical techniques essential. Runge-Kutta methods are widely used due to their simplicity and effectiveness in handling such problems. In this work, an accelerated third-order Runge-Kutta scheme is developed to efficiently approximate

solutions by incorporating time-dependent behavior directly into the formulation. The proposed method improves the accuracy of intermediate stages, leading to better overall approximations compared to standard third-order methods.

Error analysis is crucial in assessing the reliability of a numerical method. It involves examining both local truncation error and global error to determine how closely the numerical solution approximates the exact solution. For the proposed accelerated third-order Runge-Kutta method, the local truncation error is derived using Taylor series expansion, demonstrating that the leading error term is of order $O(h^4)$, which confirms third-order accuracy. Additionally, the accumulation of errors over successive steps is analyzed to ensure that the global error remains bounded. The results indicate that the improved scheme significantly reduces error magnitude compared to existing methods, particularly for non-autonomous systems where time dependence introduces additional complexity.

The order of convergence of a numerical method describes how rapidly the approximate solution approaches the exact solution as the step size decreases. A method is said to be of order p if the global error is proportional to $O(h^p)$. In this study, the proposed accelerated Runge-Kutta method is proven to be third-order convergent by satisfying the necessary order conditions derived from Taylor series expansion. Numerical experiments further validate this theoretical result by showing that halving the step size reduces the error by approximately a factor of 2^3 . This confirms that the method achieves the expected convergence rate while maintaining computational efficiency, making it suitable for solving non-autonomous ordinary differential equations.

Several physical problems in natural-science, mathematical physics and in civil-engineering are

formulated by differential equations to name a few (Dennis G. Zill, Warren S and Peter V. O'Neil). Mathematical models have constructed to identify the nature of problems; these are mostly in differential form. So for, sometimes huge number of complexity arises to handle and solve models. Many of them need a solution in a numerical way. Researchers day by day are trying to innovate many models and methods to obtain better consequences. But still huge number of work is needed to understand proper behaves of models. Models have distinct nature, but here we only talk about those models which have derivative. To get solution of these models we require approximate methods. Researchers and scholars are out on the field of innovation to hone their skills. But they haven't been able succeed properly yet. Many of them have also done a great job to construct and modified new methods. Methods have different capabilities, sometimes they are numerically and sometimes they are analytical. But every model doesn't give analytical result so in such cases we preferred numerical methods. But drawback of numerical method is that they don't give reliable result because always left some numbers called error. Few families such as RK and Euler's has sub-families, numerically. They have different converging order. Here our main focus of third converging order method. So work on third converging order, few methods tested in sense of better performance and reduce length of error. Different methods have nature separately. Some are explicit and other are implicit. Implicit is more time taken instead of explicit but it approaches quickly to the true result. Researchers day by day showing their better interest to enhance convergence, order and stability. Many equations named differential don't reach to the solution such as particular and analytical. To deal with such difficulty new innovations take place in numerical analysis. Best advantage of numerical scheme is that these have better performance than analytical.

2. MATERIALS AND METHODS

Many numerical methods have designed to attain the estimated consequences of IVP's for differential equation in nature ordinary. We can consider general function form of problems:

$$\begin{cases} y'(x) = g(x_m, y_m), & x \in [a, b] \\ y(a) = \eta \end{cases} \quad (1)$$

The closed interval $[a, b]$ is parted into a m uniform parts, $x_k = a + kh, (k = 0, 1, 2, 3, \dots, m)$, the step size is $h = x_{k+1} - x_k$. To solve the function $y(x)$ in a discrete series equidistant node $x_{m-1} < x_m$ to attend approximate values $y_{m-1} < y_m$.

Generally, function of two space variable also familiar as a derivative of the dependent variable regarding to independent has been calculated by integrating (1) from x_0 to $x_0 + h$

$$y_{m+1} = y_m + \int_{x_0}^{x_0+h} g(x_m, y_m) dx$$

Comparable form

$$y_{m+1} = y_m + h \sum_{m=1}^p p_m v_m$$

Where p_m works as a parameter and v_m works as a slope

We take p value as a 1, 2, and 3. Therefore, it adopts new form as

$$y_{m+1} = y_m + h [p_1 v_2 + p_2 v_2 + p_2 v_3] \quad (2)$$

$$v_1 = g(x_m, y_m)$$

$$v_2 = g(x_m + \alpha_2 h, y_m + h\{\beta_{21} + h\gamma_{21}g_y\}v_1)$$

$$v_3 = g(x_m + \alpha_3 h, y_m + h(\beta_{31}v_1 + \beta_{32}v_2) + h^2(\gamma_{31}v_1)g_y)$$

Now series we use which is recognize as Taylor's series to evaluate the unique value of slopes v_2 and v_3 . For two variables, Taylor's series we have:

$$\begin{aligned} T(x_m, y_m) = & y(x) + hg + h^2 \left[\frac{g_x}{2} + \frac{g_y}{2} g \right] + h^3 \left[\frac{g_{xx}}{3!} + \frac{(2g_{xy} + g_{yy}g)g}{3!} \right. \\ & \left. + \frac{(g_y g + g_x)g_y}{3!} \right] \\ & + h^4 \left[\frac{g_{xxx}}{4!} + \frac{3(g_{xxy} + g_{xyy}g)}{4!} g + \frac{(5g_y g + 3g_x)}{4!} g_{xy} + \frac{g_{yyy}}{4!} g^3 \right. \\ & \left. + \frac{4g_y g_{yy}}{4!} g^2 + \frac{3g_x g_{yy}}{4!} g + \frac{g_y^3 g}{4!} + \frac{g_x g_y^2}{4!} + \frac{g_y g_{xx}}{4!} \right] + O(h^5) \end{aligned} \quad (3)$$

Eqn. (3) is Taylor's series up to the fourth power of step size h .

3. THE ORDER-CONDITIONS OF THE METHOD

Here, in term of finding parameters suitable and reliable value we have to use Taylor's series approach. At the beginning we need to calculate OC's for the numerical integrator accessible in (2). The slopes v_2 and v_3 are taking for the purpose of expansion. We expand v_2 and v_3 by Taylor's series to achieve better expansion of functions including parameters and partial derivative. Lastly, the result of v_1, v_2 and v_3 can be inserted into (2), then equating alike terms coefficients of powers of h with (3) up to h^3 for accomplishing order-conditions which are presented below in the form of system of nonlinear equations:

$$\begin{aligned} p_1 + p_2 + p_3 &= 1 & \alpha_2 p_3 \beta_{32} &= \frac{1}{6} & \alpha_2 p_2 + \alpha_3 p_3 &= \frac{1}{2} \\ \frac{p_2^2}{2} \alpha_2^2 + \frac{p_3}{2} \alpha_3^2 &= \frac{1}{6} & p_2 \beta_{21} + p_3 (\beta_{31} + \beta_{32}) &= \frac{1}{2} & \\ \alpha_2 p_2 \beta_{21} + \alpha_3 p_3 \beta_{31} + \alpha_3 p_3 \beta_{32} &= \frac{1}{3} & p_2 \gamma_{21} + p_3 \gamma_{31} + p_3 \beta_{21} \beta_{32} &= \frac{1}{6} \end{aligned} \quad (4)$$

$$\frac{1}{2}(p_2\beta_{21}^2 + p_3\beta_{31}^2 + p_3\beta_{32}^2) + p_3\beta_{31}\beta_{32} = \frac{1}{6}$$

Above System present in Eqn (4) has eight (numerically 8) equations in nature nonlinear with 10 unknown quantities. Now we look for above system has either solution or not, if yes then we analysis it either has trivial or nontrivial solution. To clear such confusion, parameters have calculated. One of the best solution given in (4) form an explicit proposed scheme which has order third:

$$\begin{aligned} v_1 &= g(x_m, y_m) \\ v_2 &= g\left(x_m + \frac{2}{3}h, y_m + \frac{2}{3}hv_1 + 2h^2v_1g_y\right) \\ v_3 &= g\left(x_m + \frac{2h}{3}, y_m + \left\{\left(\frac{1}{24}v_1 + \frac{5}{8}v_2\right) - \frac{7}{4}hv_1g_y\right\}h\right) \\ y_{m+1} &= y_m + \frac{h}{20}(5v_1 + 7v_2 + 8v_3) \end{aligned} \quad (5)$$

This is newly developed an accelerated explicit scheme having three function evaluations per time step.

4. ERROR ANALYSIS

It is main purpose to solve ordinary differential equation numerically to attain results which are as close as possible to the exact solution. Two sources of error which affect the accuracy of numerical method named as truncation (TR) and round-off (RO). TR arises because mathematical procedures are approximated (e.g., by deserting higher-order terms), while round-off error takes the place when computers can only stock numbers with limited precision. An accuracy can totally rely on blunder what size of step size taken. Some schemes are accepted for convergence just we note how quickly reach to true point. The convergence of IVP is calculated by $e_n = |y(x_m) - y_m| < \delta$ where approximate consequence subtract from exact and A practical error bound 'δ' is a predefined tolerance level that limits how large the error in a numerical solution is allowed to be during computation. Therefore, δ relies nature and consequences of problems, it differs from 10^{-3} . The errors for numerical schemes; $Error = |y(x_m) - y_m|$.

5. ORDER OF CONVERGENCE

New accelerated proposed scheme is compared numerically and graphically with other accessible

Techniques taken from literature with accuracy of 3rd order. So, New accelerated proposed scheme is behaving numerically better than other third order existing schemes in literature. So from the numerical result we observe new accelerated proposed scheme has third order of convergent.

6. NUMERICAL EXPERIMENTS

The performance of new accelerated technique has shown in opposite of few already existing techniques present in open literature, we take few Cauchy problems and implement numerical scheme including new proposed scheme on that problems to diagnose which scheme shows better performance in all criteria of better accuracy. Different errors like as error absolute including maximum and last are acquired at the latest junction point of the interval and CPU (process time) has calculated for claiming new accelerated proposed scheme is better at all. We tested few IVP of ODE's to check the rendition of the new accelerated method in contrast to other. Two renowned techniques named in abbreviation as RK3-HM [7] and RK3-AM [5] are nominated for checking the performance in against of accelerated proposed method. MATLAB R2023a is used to analyze result and visualize its graph.

Table 1: Shows Numerically Illustration of Problem 1

| Problem 1 | | | | |
|--|-------------------------|------------|------------|------------|
| $\frac{dy}{dx} = x^2 y, \quad y(0) = 1, \quad Exact = e^{\frac{x^3}{3}}$ | | | | |
| Step-Size/ Method | | RK3-HM | RK3-AM | Proposed |
| 0.1 | Error _(Max) | 0.0031 | 0.0059 | 5.3461e-04 |
| | Error _(Last) | 0.0031 | 0.0059 | 5.3461e-04 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.05 | Error _(Max) | 8.2411e-04 | 0.0032 | 1.5434e-04 |
| | Error _(Last) | 8.2411e-04 | 0.0032 | 1.5434e-04 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.025 | Error _(Max) | 2.1194e-04 | 0.0017 | 4.1123e-05 |
| | Error _(Last) | 2.1194e-04 | 0.0017 | 4.1123e-05 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.0125 | Error _(Max) | 5.3744e-05 | 8.5624e-04 | 1.0594e-05 |
| | Error _(Last) | 5.3744e-05 | 8.5624e-04 | 1.0594e-05 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |

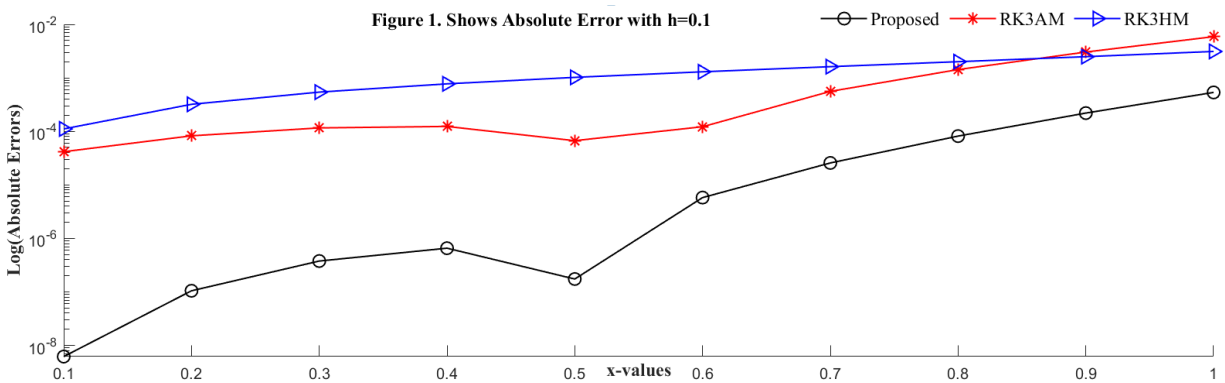


Table 2: Shows Numerically Illustration of Problem 2

| Problem 2 | | | | |
|--|-------------------------|------------|--------|------------|
| $\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1 \quad Exact = 13e^{\frac{x}{2}} - 6x - 12$ | | | | |
| Step-Size/Method | | RK3-HM | RK3-AM | Proposed |
| 0.1 | Error _(Max) | 0.0051 | 0.0336 | 0.0024 |
| | Error _(Last) | 0.0051 | 0.0336 | 0.0024 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.05 | Error _(Max) | 0.0013 | 0.0175 | 6.1948e-04 |
| | Error _(Last) | 0.0013 | 0.0175 | 6.1948e-04 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.025 | Error _(Max) | 3.2374e-04 | 0.0090 | 1.5764e-04 |
| | Error _(Last) | 3.2374e-04 | 0.0090 | 1.5764e-04 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.0125 | Error _(Max) | 8.1035e-05 | 0.0045 | 3.9750e-05 |
| | Error _(Last) | 8.1035e-05 | 0.0045 | 3.9750e-05 |

| | | | |
|----------------------|-------|-------|-------|
| Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
|----------------------|-------|-------|-------|

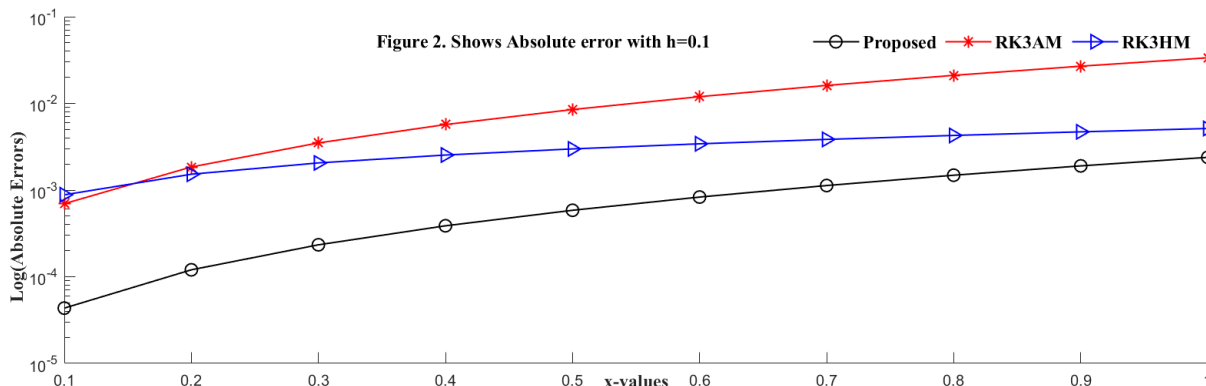
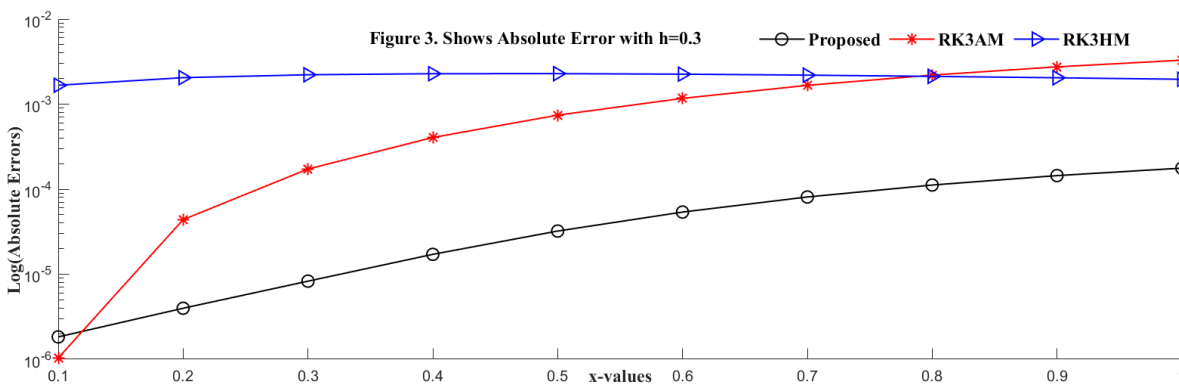


Table 3: Shows Numerically Illustration of Problem 3

| Problem 3. | | | | |
|---|-------------------------|------------|------------|------------|
| $\frac{dy}{dx} = \frac{\sin x}{y}, \quad y(0) = 1, \quad Exact = \sqrt{3 - 2 \cos x}$ | | | | |
| Step-Size/Methods | | RK3-HM | RK3-AM | Proposed |
| 0.1 | Error _(Max) | 0.0023 | 0.0033 | 1.7683e-04 |
| | Error _(Last) | 0.0020 | 0.0033 | 1.7683e-04 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.05 | Error _(Max) | 6.5681e-04 | 0.0016 | 4.5594e-05 |
| | Error _(Last) | 5.5494e-04 | 0.0016 | 4.5594e-05 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.025 | Error _(Max) | 1.8632e-04 | 8.2142e-04 | 1.1591e-05 |
| | Error _(Last) | 1.5560e-04 | 8.2142e-04 | 1.1591e-05 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |
| 0.0125 | Error _(Max) | 5.2173e-05 | 4.1013e-04 | 2.9229e-06 |
| | Error _(Last) | 4.3185e-05 | 4.1013e-04 | 2.9229e-06 |
| | Time _(PT) | 0.0E0 | 0.0E0 | 0.0E0 |



In Fig (1-3) we display the graphs of three numerical schemes among them two well-known

schemes named in abbreviation as RK3HM [7] and RK3AM [5], and one new accelerated

proposed Scheme. In every figure, x-values are spotted in x-axis and Log (Absolute Error) in y-axis. Among all graphically results the graph of new accelerated proposed scheme shows less error than other scheme whose graphical result present in graph. It is clearly observed that new an accelerated proposed scheme is better numerically and converges rapidly in comparison with RK3HM and RK3AM to the exact solution.

7. CONCLUSION

This paper has designed an accelerated third order Runge Kutta Method having partial derivative of the function $g(x, y)$ with respect to dependent variable y in function evaluations. Numerically and graphically, proposed scheme is examined for better to better consequences. Consequences (graphically and numerically) gives evidence that proposed scheme is more accurate, converge rapidly and competent than other in term of approximations. Absolute Errors (last and maximum) with CPU time is numerically obtained by using altered step size for comparison with existing schemes. The RK3HM [7] and RK3AM [5] rapidly grows in error than proposed one. According to the numerical result, newly developed accelerated method has accuracy of order three and keep nature explicit. We finally on that result the accelerated proposed method is more capable and appropriate.

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