Spectrum of Engineering Sciences
ISSN (e) 3007-3138 (p) 3007-312X
Volume 4, Issue 4, 2026

DIAMOND OSTROWSKI TYPE INEQUALITIES ON TIME SCALES

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Keywords
Diamond- calculus, Ostrowski inequality, The weighted Ostrowski inequalities, Time scales

Article History
Received on 29 March, 2026
Accepted on 25 April, 2026
Published on 27 April, 2026

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Abstract

This article explores Ostrowski-type inequalities within the framework of time scales, empowering the theory of Diamond- derivatives. The study establishes the validity of these inequalities in both nabla and delta calculus, highlighting a unified approach applicable to arbitrary dynamic time scales. Furthermore, it extends these results to enclose specialized inequalities as unique cases for specific time scales, such as and . These findings not only bridge discrete and continuous analyses but also upgrade the mathematical tools for dynamic systems on generalized time scales, offering broad applications in pure and applied mathematics.

1. Introduction:

The progress in mathematical inequalities has continued to grow rapidly since nineteenth century. Holder, Minkowski, Hadamard, Hardy, Schwartz, and Cauchy are famous mathematicians known for their work in inequalities. To highlight the History of inequalities, we quote from [5]. A Ukrainian Mathematician, Alexandar Markowich Ostrowski , in 1938 discovered an interesting integral inequality called Ostrowski inequality in [8]. Ostrowski type inequalities are used to estimate the absolute deviation of functional value from integral mean. It also has powerful applications in operator theory, statistics, special means, and integral numerical methods. Ostrowski inequality is derived by using Montgomery identity. The famous Montgomery identity in the original form can be seen in [9, p-565]. A generalization of Ostrowski inequality has been established by introducing weighted integrals [9,11]. Integration of function with weight is used in many mathematical problems like spectral analysis, approximation theory and statistical analysis [19].

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [5] in 1988. Nowadays, most of the development of time scale theory

is based on the unification of discrete and continuous analytical methods. At this point, the effectiveness of various dynamic derivative formulae, such as the standard Δ (delta) and ∇ (nabla) derivatives, in approximating functions and solutions of non-linear differential equations have been investigated in [1,5]. It has been seen in many research papers such as [6,7] in which it is suggested that the Diamond- α derivative is defined as a convex combination, of the Δ and the ∇ dynamic derivatives and gives an exact approximation to the conventional derivative. In 2006, Henderson, Sheng, Davis and Fadag, were the first mathematicians who combined $\Diamond_{\alpha}$ dynamic integral (resp. derivative) as a linear combination of the ∇ and Δ dynamic integral (resp. derivative) on time scales [7]. From the perspective of computational applications diamond- α derivatives offer better approximations. In [3], Thomas Matthews and Martin Bohner established Ostrowski inequalities on time scales using only delta calculus. The present article aims to generalize some results from Mathematical Analysis known as Montgomery identity and Ostrowski inequality by using diamond- α calculus on arbitrary time scales.

This manuscript is organized as follows. In Section 1, we present the important notions of (∇ and Δ) and diamond- α calculus on time scales. It also provides a brief survey of literature on Ostrowski inequalities and their weighted version developed on arbitrary time scales. Section 2 is dedicated to our main results. In this part, we first established diamond- α Montgomery identity on time scales and utilized it to drive diamond α Ostrowski inequality and their weighted versions on generalized time scales. The results are new and important for the further study of Ostrowski and Ostrowski-type inequalities on diamond- α calculus. Throughout this article, we take over all considerable integrals exist and finite, and time scale is denoted by $\mathbb{T}$, therefore, $l_a, l_b \in \mathbb{T}$, with $l_a < l_b$ and $[l_a, l_b]_{\mathbb{T}} = [l_a, l_b] \cap \mathbb{T}$.

2. Preliminaries:

A time scale $\mathbb{T}$ is defined to be an arbitrary closed subset of the real numbers $\mathbb{R}$, with the standard inherited topology. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $\rho(t) := \sup\{s \in$

$\mathbb{T}: s < t\}$ respectively, where $\inf\phi = \sup\mathbb{T}$ and $\sup\phi = \inf\Gamma$.

The forward and backward graininess functions are defined by the mappings: $\mu: \mathbb{T} \longrightarrow [0, \infty)$ and $v: \mathbb{T} \longrightarrow [0, \infty)$ respectively as

$$\mu := \sigma(t) - t, v := t - \rho(t)$$

for $t \in \mathbb{T}$. The delta derivative of the function $\pi: \mathbb{T} \rightarrow \mathbb{R}$ denoted by π^{Δ}. The nabla derivative of π denoted by π^{∇}. The symbol C_{rd} or $C_{\mathrm{rd}}(\mathbb{T})$ or $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and the symbol C_{ld} or $C_{\mathrm{ld}}(\mathbb{T})$ or $C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R})$ defined to be the set of rd-continuous and ld-continuous functions respectively.

Furthermore, the basic notions of delta and nabla anti derivatives with their properties and details can be seen in [1,2], although we state some important results here which we used in our main results.

Theorem 2.1. Every regulated function on a compact interval is bounded.

The next result is presented in [1, 2, 18] which shows the equivalence of delta and nebla integrals.

Theorem 2.2. If the function $\pi: \mathbb{T} \rightarrow \mathbb{R}$ is continuous, then for all $l_a, l_b \in \mathbb{T}$ with $l_a < l_b$ we have

$$\int_{l_a}^{l_b} \pi(t)\Delta t = \int_{l_a}^{l_b} \pi(\rho(t))\nabla t \text{ and } \int_{l_a}^{l_b} \pi(t)\nabla t = \int_{l}^{l}$$

The following useful notions are also used in this article.

Definition 2.3. For $t, \theta \in \mathbb{T}$ define the functions

$$h_0(t, \theta) = g_0(t, \theta) = \hat{h}_0(t, \theta) = \hat{g}_0(t, \theta) \equiv 1$$

and the recursive relations for $h_n, g_n, \hat{h}_n, \hat{g}_n$ are given as follows

$$h_{n+1}(t, \theta) = \int_{\theta}^{t} h_n(\Phi, \theta)\Delta\Phi, \quad g_{n+1}(t, \theta) = \int_{\theta}^{t} g_n($$

$$\hat{h}_{n+1}(t, \theta) = \int_{\theta}^{t} \hat{h}_n(\Phi, \theta)\nabla\Phi, \quad \hat{g}_{n+1}(t, \theta) = \int_{\theta}^{t} \hat{g}_n($$

for $n \in \mathbb{N}_0$.

Theorem 2.4. The functions g_n and h_n satisfy the relationship

$$h_n(t, \theta) = (-1)^n g_n(\theta, t)$$

for all $t \in \mathbb{T}$ and all $\theta \in \mathbb{T}^{\kappa^n}$.

2.1. Diamond-Alpha Calculus

For a detailed introduction of the diamond- α calculus on time scales we refer [6, 7, 12]. In this section, we give properties and related notions of $\Diamond_{\alpha}$ calculus.

Definition 2.5. If π is both nabla and delta differentiable at $t \in \mathbb{T}$, then π is Diamond-alpha differentiable at t and

$$\pi^{\Diamond\alpha}(t) = \alpha\pi^{\Delta}(t) + (1 - \alpha)\pi^{\nabla}(t), 0 \leqslant \alpha \leqslant 1$$

Definition 2.6. Let $l_a, l_b \in \mathbb{T}$ with $l_a < l_b$, and $\pi: \mathbb{T} \rightarrow \mathbb{R}$. Then, the Diamond- α integral from l_a to l_b of the function π is given by

$$\int_{l_a}^{l_b} \pi(t)\Diamond_{\alpha}t = \alpha\int_{l_a}^{l_b} \pi(t)\Delta t + (1 - \alpha)\int_{l_a}^{l_b} \pi(t)\nabla t, 0 \leq \alpha \leq 1, (2$$

if there exists nabla and delta integrals of π on $\mathbb{T}$.

Remark 2.7. If $\alpha = 1$, then the diamond- α derivative and diamond- α integral reduces to the delta derivative and delta integral respectively. If $\alpha = 0$, then the diamond- α derivative and diamond- α integral coincides with the nabla derivative and nabla integral respectively.

Remark 2.8. If we take $\mathbb{T} = \mathbb{R}$, we get

$$\int_{l_a}^{l_b} \pi(t)\Diamond_{\alpha}t = \int_{l_a}^{l_b} \pi(t)dt, \text{ where } a, b \in \mathbb{R}$$

and for $\mathbb{T} = \mathbb{Z}$, we obtain

$$\int_{l_a}^{l_b} \pi(t)\Diamond_{\alpha}t = \sum_{t=l_a}^{l_b-1} [\alpha\pi(t) + (1 - \alpha)\pi(t + 1)]$$

Theorem 2.9. Take $\alpha \in [0,1]$ and $g, \pi \in C(\mathbb{T}, \mathbb{R})$. If $l_a, l_b, l_c \in \mathbb{T}$ with $l_a \leq l_c \leq l_b$ and $\beta \in \mathbb{R}$, then

1. $\int_{l_a}^{l_b}[\pi(t) + g(t)]\Diamond_{\alpha}t = \int_{l_a}^{l_b}\pi(t)\Diamond_{\alpha}t + \int_{l_a}^{l_b} g(t)\Diamond_{\alpha}t$
2. $\int_{l_a}^{l_b}(\beta f)(t)\Diamond_{\alpha}t = \beta\int_{l_a}^{l_b}\pi(t)\Diamond_{\alpha}t$
3. $\int_{l_a}^{l_b}\pi(t)\Diamond_{\alpha}t = -\int_{l_b}^{l_a}\pi(t)\Diamond_{\alpha}t$

4. $\int_{l_a}^{l_b}\pi(t)\Diamond_\alpha t=\int_{l_a}^{l_c}\pi(t)\Diamond_\alpha t+\int_{l_c}^{l_b}\pi(t)\Diamond_\alpha t$

5. $\int_{l_a}^{l_a}\pi(t)\Diamond_\alpha t=0$

6. $\pi(t)\geq 0$ for all $t\in[l_a,l_b]$, then $\int_{l_a}^{l_b}\pi(t)\Diamond_\alpha t\geq 0$

7. $\pi(t)\leq g(t)$ for all $t\in[l_a,l_b]$, then $\int_{l_a}^{l_b}\pi(t)\Diamond_\alpha t\leq\int_{l_a}^{l_b}g(t)\Diamond_\alpha t$;

8. $\left|\int_{l_a}^{l_b}\pi(t)\Diamond_\alpha t\right|\leq\int_{l_a}^{l_b}|\pi(t)|\Diamond_\alpha t$

Theorem 2.10. (Diamond- α Hölder inequality) If $g,u:\mathbb{T}\to\mathbb{R}$ are $\Diamond_\alpha$-integrable on $[l_a,l_b]_{\mathbb{T}}$, then

$$\int_{l_a}^{l_b}|u(t)g(t)|\Diamond_\alpha t\leq\left(\int_{l_a}^{l_b}|u(t)|^p\Diamond_\alpha t\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b}|g(t)\right.$$

where $p>1$ and $q=\frac{p}{p-1}$.

Definition 2.11. Take $\alpha\in[0,1]$ and $\mathbb{T}$ be a time scale, for $\theta,t\in\mathbb{T}$ and $n\in\mathbb{N}^0$ define the functions

$$\tilde{h}_n(t,\theta):=\alpha h_n(t,\theta)+(1-\alpha)\hat{h}_n(t,\theta)\quad(2.1.3)$$
$$\tilde{g}_n(t,\theta):=\alpha g_n(t,\theta)+(1-\alpha)\hat{g}_n(t,\theta)\quad(2.1.4)$$

2.2. Ostrowski Inequality and Montgomery Identity

The classical Ostrowski's inequality presented in [11], states that for all $t\in[l_a,l_b]$, if derivative π' exists and is bounded on (l_a,l_b), then

$$\left|\int_{l_a}^{l_b}\pi(t)dt-\pi(t)(l_b-l_a)\right|\leq\left[\left(t-\frac{l_a+l_b}{2}\right)^2+\frac{1}{4}(\right.$$

where

$$\|\pi'\|_\infty:=\sup_{t\in(l_a,l_b)}|\pi'(t)|<\infty$$

The classical Montgomery identity in [9] states that if $\pi:[l_a,l_b]\to\mathbb{R}$ be differentiable on $[l_a,l_b]$, and $\pi':[l_a,l_b]\to\mathbb{R}$ be integrable on $[l_a,l_b]$, then the following identity holds

$$\pi(t)=\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi(t)d\theta+\frac{1}{l_b-l_a}\int_{l_a}^{l_b}p(t,\theta)\pi'(\theta$$

where

$$p(t,\theta)=\begin{cases}\theta-l_a, & l_a\leq\theta<t\\ \theta-l_b, & t\leq\theta\leq l_b\end{cases}$$

In 2008, M. Bohner and T. Methews in [3] developed the Montgomery identity using only delta calculus on arbitrary time scales and utilized it to establish Ostrowski and weighted Ostrowski inequalities on time scales. The corresponding discrete version of these inequalities has also been discussed in the same article.

3. The Dynamic Ostrowski's Inequality

In this section, we drive Montgomery identity and Ostrowski Inequality along with a weighted version on generalized time scales by using only nabla derivatives (resp. integral). Furthermore, we extend these dynamic inequalities by using diamond- α derivatives (resp. integral), which is a convex combination of delta and nabla derivatives (resp. integral) on time scales.

3.1. Montgomery Identity

Theorem 3.1. (Nabla Montgomery Identity). Let $l_a,l_b,\theta,t\in\mathbb{T},l_a<l_b$ and $\pi:[l_a,l_b]_{\mathbb{T}}\to\mathbb{R}$ be nabla differentiable. Then

$$\pi(t)=\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi^\rho(\theta)\nabla\theta+\frac{1}{l_b-l_a}\int_{l_a}^{l_b}p(t,\theta)\pi^\nabla(\theta)\nabla\theta\quad(3.1.1)$$

where

$$p(t,\theta)=\begin{cases}\theta-l_a, & l_a\leq\theta<t\\ \theta-l_b, & t\leq\theta\leq l_b\end{cases}$$

Proof. Using integration by parts formula for nabla integrals [1,2], for the interval $[\,l_a,t\,]$, we have

$$\int_{l_a}^{t}(\theta-l_a)\pi^\nabla(\theta)\nabla\theta$$
$$=(t-l_a)\pi(t)-\int_{l_a}^{t}\pi^\rho(\theta)\nabla\theta$$

and similarly for $[t,l_b]$, we get

$$\int_{t}^{l_b}(\theta-l_b)\pi^\nabla(\theta)\nabla\theta$$
$$=(l_b-t)\pi(t)-\int_{t}^{l_b}\pi^\rho(\theta)\nabla\theta$$

Therefore

$$\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi^{\rho}(\theta)\nabla\theta+\frac{1}{l_b-l_a}\int_{l_a}^{l_b}p(t,\theta)\pi^{\nabla}(\theta)\nabla\theta$$

$$=\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi^{\rho}(\theta)\nabla\theta+\frac{1}{l_b-l_a}\Big[(l_b-l_a)\pi(t)-$$

$$=\pi(t)$$

Remark 3.2. (Continuous version) Taking $\mathbb{T}=\mathbb{R}$., we get

$$\pi(t)=\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi(\theta)d\theta+\frac{1}{l_b-l_a}\int_{l_a}^{l_b}p(t,\theta)\pi'(\theta)d\theta$$

The above result is known as the classical version of Montgomery identity. (see [9])

Remark 3.3. (Discrete version) Taking $\mathbb{T}=\mathbb{Z}$. Let $l_a=0, l_b=n, \theta=s, t=t$ and $\pi(k)=x_k$ then

$$x_t=\frac{1}{n}\sum_{s=1}^{n}p(t,s)\nabla x_s+\frac{1}{n}\sum_{s=1}^{n}x_{s-1}\qquad(3.1.2)$$

$$x_t=\frac{1}{n}\sum_{s=1}^{n}p(t,s)\nabla x_s+\frac{1}{n}\sum_{s=0}^{n-1}x_s\qquad(3.1.3)$$

where

$$p(t,0)=$$

$$p(1,s)=s-n \text{ for } 1$$

$$p(n,s)=s \text{ for } 0\leq$$

$$p(t,s)=\begin{cases}s, & 0\leq s<t\\ s-n, & t\leq s\leq n-1.\end{cases}$$

3.2. Ostrowski Inequalities

Theorem 3.4. (Nabla Ostrowski inequality). Let $l_a,l_b,\theta,t\in\mathbb{T}, l_a<l_b$ and $\pi:[l_a,l_b]_{\mathbb{T}}\to\mathbb{R}$ be nabla differentiable. Then

$$\left|\pi(t)-\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi^{\rho}(\theta)\nabla\theta\right|\leq\frac{M}{l_b-l_a}\Big(\hat{h}_2(t,l_a)+\hat{}$$

where

$$M=\sup_{l_a<t<l_b}\left|\pi^{\nabla}(t)\right|$$

and $\hat{h}_2$ is defined in remark (2.3).

Proof. By Montgomery identity (Theorem 3.1), we have

$$\left|\pi(t)-\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi^{\rho}(\theta)\nabla\theta\right|=\left|\frac{1}{l_b-l_a}\int_{l_a}^{l_b}p(t,\theta)\pi^{\nabla}(\theta)\nabla\theta\right|$$

$$\leq\frac{M}{l_b-l_a}\left(\int_{l_a}^{t}|\theta-l_a|\nabla\theta+\int_{t}^{l_b}|\theta-l_b|\nabla\theta\right)$$

$$=\left(\int_{l_a}^{t}(\theta-l_a)\nabla\theta+\int_{t}^{l_b}(\theta-l_b)\nabla\theta\right)\frac{M}{l_b-l_a}$$

$$=\left(\hat{h}_2(t,l_a)+\hat{h}_2(t,l_b)\right)\frac{M}{l_b-l_a}$$

□

Remark 3.5. (Continuous version) Taking $\mathbb{T}=\mathbb{R}$, we get

$$\left|\pi(t)-\frac{1}{l_b-l_a}\int_{l_a}^{l_b}\pi(\theta)d\theta\right|\leq M(l_b-l_a)\left[\frac{\left(\theta-\frac{l_a+l_b}{2}\right)^2}{(l_b-l_a)^2}+\frac{1}{4}\right](3.2$$

where

$$M=\sup_{l_a<t<l_b}\left|\pi'(t)\right|$$

Which coincides with classical Ostrowski inequality (see [11]). □

Remark 3.6. (Discrete version) Taking $\mathbb{T}=\mathbb{Z}$. Let $l_a=0, l_b=n, \theta=s, t=t$ and $\pi(k)=x_k$. Then by using (1.24), we get

$$\left|x_t-\frac{1}{n}\sum_{s=1}^{n}x_s\right|\leq\frac{M}{l_b-l_a}\left(\hat{h}_2(t,l_a)+\hat{h}_2(t,l_b)\right)$$

$$=\frac{M}{n}\left(\sum_{s=1}^{t}s+\sum_{s=t+1}^{n}(s-n)\right)$$

$$=\frac{M}{n}\left(\frac{1}{2}(t-n)(t-n+1)+\frac{1}{2}t(t+1)\right)$$

$$=\frac{M}{n}\left[\left(t-\frac{n-1}{2}\right)^2+\frac{n^2-1}{4}\right]$$

Theorem 3.7. (Nabla Weighted Ostrowski Inequality). Let $l_a,l_b,\theta,t,\tau\in\mathbb{T}, l_a<l_b$ and $\pi:[l_a,l_b]_{\mathbb{T}}\to\mathbb{R}$ be differentiable, $q\in C_{\mathrm{ld}}(\mathbb{T})$. Then

$$\left|\pi(t)-\int_{l_a}^{l_b}q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right|\leq\int_{l_a}^{l_b}q^{\rho}(\theta)|\rho(\theta)-t|M\nabla\theta,(3.2.4)$$

where

$$M = \sup_{l_a<\tau\le\rho(l_b)} \left|\pi^{\nabla}(\tau)\right|$$

and

$$\int_{l_a}^{l_b} q^{\rho}(\theta)\nabla\theta = 1, q(\theta) \ge 0$$

Proof. We have

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right|$$

$$= \left|\int_{l_a}^{l_b} q^{\rho}(\theta)\big(\pi(t) - \pi^{\rho}(\theta)\big)\nabla\theta\right|,$$

by property of nabla integral, we get

$$\le \int_{l_a}^{t} q^{\rho}(\theta)|\pi(t) - \pi^{\rho}(\theta)|\nabla\theta$$

$$+ \int_{t}^{l_b} q^{\rho}(\theta)|\pi(t) - \pi^{\rho}(\theta)|\nabla\theta$$

$$\le \int_{l_a}^{t} q^{\rho}(\theta) \int_{\rho(\theta)}^{t} \left|\pi^{\nabla}(\tau)\right| \nabla\tau\nabla\theta$$

$$+ \int_{t}^{l_b} q^{\rho}(\theta) \int_{t}^{\rho(\theta)} \left|\pi^{\nabla}(\tau)\right| \nabla\tau\nabla$$

thus

$$\le M \int_{l_a}^{l_b} q^{\rho}(\theta)|\rho(\theta) - t|\nabla\theta \qquad (3.2.5)$$

Remark 3.8. (Continuous version). Taking $\mathbb{T} = \mathbb{R}$. Then, we get

$$\left|\pi(t) - \int_{l_a}^{l_b} q(\theta)\pi(\theta)d\theta\right| \le M \int_{l_a}^{l_b} q(\theta)|\theta - t|d\theta(3$$

and $\int_{l_a}^{l_b} q(\theta)d\theta = 1, q(\theta) \ge 0$.

The above inequality is a classical version of the weighted Ostrowski inequality (see [13]).

Remark 3.9. (Discrete version). Taking $\mathbb{T} = \mathbb{Z}$. Let $l_a = 0, l_b = n, \theta = s, t = t, \tau = k$ and $\pi(k) = x_k$. Then $\sum_{t=1}^{n} q_t = 1, q_t \ge 0$ and by (1.24), we get

$$\left|x_i - \sum_{s=1}^{n} q_s x_s\right| \le M \sum_{s=1}^{n} q_s|s - t| \qquad (3.2.7)$$

This is the Discrete version of weighted Ostrowski inequality.

Corollary 3.10. Let $l_a, l_b, \theta, t, \tau \in \mathbb{T}, l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$ be nabla differentiable, $q \in C_{ld}(\mathbb{T})$. Then for $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, we have

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right| \le M\left(\int_{l_a}^{l_b} |\rho(\theta) - t|^{p}\nabla\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} |q^{\rho}($$

where M and weight function q are same as in Theorem 3.7.

Proof. Therefore $q^{\rho}(\theta) \ge 0$. We have

$$\left|\pi(t) - \int_{l_a}^{l_b} \pi^{\rho}(\theta)q^{\rho}(\theta)\nabla\theta\right| \le \int_{l_a}^{l_b} q^{\rho}(\theta)|\rho(\theta) - t|M\nabla\theta$$

$$= \int_{l_a}^{l_b} |q^{\rho}(\theta)||\rho(\theta) - t|\nabla\theta$$

hence, by using Holder's inequality on Nebla integrals, we get

$$\le M\left(\int_{l_a}^{l_b} |\rho(\theta) - \theta|^{p}\nabla\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} \left(q^{\rho}(\theta)\right)^{q}\nabla\theta\right)^{\frac{1}{q}}$$

Remark 3.11. (Continuous version). Taking $\mathbb{T} = \mathbb{R}$. Then $\int_{l_a}^{l_b} q(\theta)d\theta = 1, q(\theta) \ge 0$, then for $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, and by (1.21) we get

$$\left|\pi(t) - \int_{l_a}^{l_b} q(\theta)\pi(\theta)d\theta\right| \le M\left(\int_{l_a}^{l_b} |\theta - t|^{p}d\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} (q(\theta))^{q}d\theta$$

Remark 3.12. (Discrete version). Taking $\mathbb{T} = \mathbb{Z}$. Let $l_a = 0, l_b = n, \theta = s, t = t, \tau = k$ and $\pi(k) = x_k$. Then $\sum_{t=1}^{n} q_t = 1, q_t \ge 0$ and by (1.24), we get

$$\left|x_i - \sum_{s=1}^{n} q_s x_s\right| \le M\left(\sum_{s=1}^{n} q_s^{q}\right)^{\frac{1}{q}}\left(\sum_{s=1}^{n} |s - t|^{p}\right)^{\frac{1}{p}} (3.2.10)$$

for

$$\frac{1}{p} + \frac{1}{q} = 1, p > 1$$

Corollary 3.13. Let $l_a, l_b, \theta, t, \tau \in \mathbb{T}, l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$ be nabla differentiable, $q \in C_{ld}(\mathbb{T})$. Then

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right| \le M \sup_{l_a<\theta\le l_b} q^{\rho}(\theta)[\hat{g}_2(t, l_a) + \hat{g}_2(t, l_b)$$

where where M and weight function q are same as in Theorem 3.7.

Proof. we have

$$\left|\pi(t)-\int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right| \leq M\int_{l_a}^{l_b} q^{\rho}(\theta)|t-\rho(\theta)|\nabla\theta$$

so the integral on the right side of this inequality can be written as

$$\int_{l_a}^{l_b} q^{\rho}(\theta)|\rho(\theta)-\theta|\nabla\theta \leq \sup_{l_a<\theta\leq l_b} q^{\rho}(\theta)\Bigg(\int_{l_a}^{t} (t-\rho($$

$$= \sup_{l_a<\theta\leq l_b} q^{\rho}(\theta)\Bigg(\int_{t}^{l_a} (\rho(\theta) -$$

$$= \sup_{l_a<\theta\leq l_b} q^{\rho}(\theta)[\hat{g}_2(l_a,t)+\hat{g}$$

Remark 3.14. (Continuous version). Taking $\mathbb{T}=\mathbb{R}$. Then $\int_{l_a}^{l_b} q(\theta)d\theta=1, q(\theta)\geq 0$, by (1.21), we get

$$\left|\pi(t)-\int_{l_a}^{l_b} q(\theta)\pi(\theta)d\theta\right| \leq M\sup_{l_a<\theta\leq l_b} q(\theta)\Bigg[\frac{\left(\theta-\frac{l_a+}{2}\right.}{(l_b-l_c}$$

Remark 3.15. (Discrete version). Taking $\mathbb{T}=\mathbb{Z}$. Let $l_a=0, l_b=n, \theta=s, t=t, \tau=k$ and $\pi(k)=x_k$. Then $\sum_{t=1}^{n} q_t=1, q_t\geq 0$ and by (1.24), we get

$$\left|x_i-\sum_{s=1}^{n} q_s x_s\right| \leq M\left[\left(t-\frac{n+1}{2}\right)^2 + \frac{n^2-1}{4}\right]\max_{s=1\ldots n} q(s),$$

where

$$M=\max_{1\leq k\leq n-1}|\nabla x_k|$$

Corollary 3.16. Let $l_a, l_b, \theta, t, \tau\in\mathbb{T}, l_a<l_b$ and $\pi:[l_a,l_b]_{\mathbb{T}}\to\mathbb{R}$ be nabla differentiable, $q\in C_{ld}(\mathbb{T})$. Then

$$\left|\pi(t)-\int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right| \leq M\Bigg(\frac{\rho(l_b)-l_a}{2}+\Big|t-$$

where M and weight function q are same as in Theorem 3.7.

3.3. Diamond- α Montgomery Identity

Theorem 3.17. [Montgomery Identity] Consider $\alpha\in[0,1]$ and $l_a, l_b, \theta, t\in\mathbb{T}, l_a<l_b$ and $\pi:[l_a,l_b]_{\mathbb{T}}\to\mathbb{R}$. Let π^{Δ} and π^{∇} exist and continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then

$$\pi(t)=\frac{1}{[(\alpha)^2+(1-\alpha)^2](l_b-l_a)}\Bigg[\int_{l_a}^{l_b}\pi^{\Diamond_\alpha}(\theta)P(\theta,t)\Diamond_\alpha\theta+\alpha^2\int$$

$$+(1-\alpha)^2\int_{l_a}^{l_b}\pi^{\rho}(\theta)\nabla\theta\Bigg]-\alpha(1-\alpha)G(\theta,t)$$

where

$$p(t,\theta)=\begin{cases}\theta-l_a, & l_a\leq\theta<t\\ \theta-l_b, & t\leq\theta\leq l_b\end{cases}$$

$$G(\theta,t)=\Bigg[(l_b-l_a)\Big(\pi\big(\sigma(t)\big)+\pi\big(\rho(t)\big)\Big)-\int_{l_a}^{l_b}\pi(\theta)\nabla\theta-\int_{l_a}^{l_b}\pi($$

Proof. Using definitions (2.5 and 2.6), we have

$$\int_{l_a}^{t}(\theta-l_a)\pi^{\Diamond_\alpha}(\theta)\Diamond_\alpha\theta=\alpha^2\int_{l_a}^{t}(\theta-l_a)\pi^{\Delta}(\theta)\Delta\theta+(1-\alpha)^2\int_{l_a}^{t}$$

$$+\alpha(1-\alpha)\Bigg[\int_{l_a}^{t}(\theta-l_a)\pi^{\Delta}(\theta)\nabla\theta+\int_{l_a}^{t}$$

Using integration by parts formula for nabla and delta integrals [1,2] and definition 2.6, the righthand side of the above expression can be written as

$$=\alpha^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}\pi^{\sigma}(\theta)\Delta\theta\Bigg]+(1-\alpha)^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}$$

$$+\alpha(1-\alpha)\Bigg[\int_{l_a}^{t}(\theta-l_a)\pi^{\nabla}\big(\sigma(\theta)\big)\nabla\theta+\int_{l_a}^{t}(\theta-l_a)\pi^{\Delta}\big(\rho(\theta)\big)\Delta$$

$$=\alpha^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}\pi^{\sigma}(\theta)\Delta\theta\Bigg]+(1-\alpha)^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}$$

$$+\alpha(1-\alpha)\Bigg[(t-l_a)\pi\big(\sigma(t)\big)-\int_{l_a}^{t}\pi\Big(\rho\big(\sigma(\theta)\big)\nabla\theta+(t-l_a)\pi($$

$$=\alpha^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}\pi^{\sigma}(\theta)\Delta\theta\Bigg]+(1-\alpha)^2\Bigg[(t-l_a)\pi(t)-\int_{l_a}^{t}$$

$$+\alpha(1-\alpha)\Bigg[(t-l_a)\Big(\pi\big(\sigma(t)\big)+\pi\big(\rho(t)\big)\Big)-\int_{l_a}^{t}\pi(\theta)\nabla\theta-\int_{l_a}^{t}$$

Similarly,

$$\int_{l_a}^{t} (\theta - l_b)\pi^{\Diamond_\alpha}(\theta)\Diamond_\alpha\theta = \alpha^2 \Big[(l_b - t)\pi(t) - \int_{t}^{l_b} \pi^{\sigma}$$

$$+ \alpha(1-\alpha)\Big[(l_b - t)\big(\pi(\sigma$$

Finally,

$$\int_{l_a}^{l_b} \pi^{\Diamond_\alpha}(\theta)P(\theta,t)\Diamond_\alpha\theta = \alpha^2 \Big[(l_b - l_a)\pi(t) - \int_{l_a}^{l_b} \pi^{\sigma}$$

$$+ \alpha(1-\alpha)G(\theta,t)$$

which completes the proof.

Remark 3.18. When $\alpha = 1$, we get

$$\pi(t) = \frac{1}{l_b - l_a}\int_{l_a}^{l_b} \pi^{\sigma}(\theta)\Delta\theta + \frac{1}{l_b - l_a}\int_{l_a}^{l_b} p(t,\theta)\pi^{\Delta}(\theta)\Delta\theta$$

which is the Montgomery identity developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$\pi(t) = \frac{1}{l_b - l_a}\int_{l_a}^{l_b} \pi^{\rho}(\theta)\nabla\theta + \frac{1}{l_b - l_a}\int_{l_a}^{l_b} \pi^{\nabla}(\theta)p(t,\theta)\nabla\theta$$

which is the Montgomery identity developed using nabla calculus (see (3.1.1)).

3.4. Diamond- α Ostrowski inequality via Diamond- α Montgomery identity

Theorem 3.19. Consider $\alpha \in [0,1]$ and $l_a, l_b, \theta, t \in \mathbb{T}, l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$. Let π^{Δ} and π^{∇} exist and continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then

$$\Big|\pi(t)[(\alpha)^2 + (1-\alpha)^2] - \frac{1}{(l_b - l_a)}\Big[\alpha^2 \int_{l_a}^{l_b} \pi^{\sigma}(\theta)$$

$$- \alpha(1-\alpha)G(\theta,t)]\Big| \leq \frac{M}{l_b - l_a}\big(\tilde{h}_2(t,l_a) + \tilde{h}_2(t,l$$

where $G(\theta,t)$ is same as in Theorem 3.17 and

$$M = \sup_{l_a<t<l_b} \left|\pi^{\Diamond}(t)\right| \qquad (3.4.1)$$

Proof. Using diamond- α Montgomery identity, we have

$$|\pi(t)[(\alpha)^2 + (1-\alpha)^2] - \frac{1}{(l_b - l_a)}\Big[\alpha^2 \int_{l_a}^{l_b} \pi^{\sigma}(\theta)\Delta\theta + (1-\alpha)^2 \int$$

$$+\alpha(1-\alpha)G(\theta,t)| = \left|\frac{1}{l_b - l_a}\int_{l_a}^{l_b} p(t,\theta)\pi^{\Diamond_\alpha}(\theta)\Diamond_\alpha\theta\right|$$

$$\leq \frac{M}{l_b - l_a}\bigg(\int_{l_a}^{t} |\theta - l_a|\Diamond_\alpha\theta + \int_{t}^{l_b} |\theta - l_b|$$

$$= \frac{M}{l_b - l_a}\bigg(\int_{l_a}^{t} (\theta - l_a)\Diamond_\alpha\theta + \int_{l_b}^{t} (\theta - l_b)$$

$$= \frac{M}{l_b - l_a}\big(\tilde{h}_2(t,l_a) + \tilde{h}_2(t,l_b)\big)$$

Remark 3.20. When $\alpha = 1$, we get

$$\left|\pi(t) - \frac{1}{(l_b - l_a)}\int_{l_a}^{l_b} \pi^{\sigma}(\theta)\Delta\theta\right| \leq \big(h_2(t,l_a) + h_2(t,l_b)\big)\frac{M}{l_b - l_a},$$

which is Ostrowski inequality developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$\left|\pi(t) - \frac{1}{(l_b - l_a)}\int_{l_a}^{l_b} \pi^{\rho}(\theta)\nabla\theta\right| \leq \big(\hat{h}_2(t,l_a) + \hat{h}_2(t,l_b)\big)\frac{M}{l_b - l_a},$$

which is Ostrowski inequality developed using nabla calculus [see (3.2.1)].

3.5. Diamond- α Weighted Ostrowski inequality

Theorem 3.21. Assume $l_a, l_b, \theta, t, \tau \in \mathbb{T}$ with $l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$. Let π^{∇_α} exist and corresponding $\pi^{\Delta}, \pi^{\nabla}$ are continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then the following inequality

$$\bigg|\pi(t) - \bigg(\alpha \int_{l_a}^{l_b} \pi^{\sigma}(\theta)q^{\sigma}(\theta)\Diamond_\alpha\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\Diamond_\alpha\theta$$

$$\leq M\bigg(\alpha \int_{l_a}^{l_b} q^{\sigma}|\sigma(\theta) - t|\Diamond_\alpha\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}|\rho(\theta) - t|\Diamond_\alpha\theta\bigg)$$

will hold for a nonnegative continuous function $q: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$ with the property that

$$\int_{l_a}^{l_b} q^{\sigma}(\theta)\Diamond_\alpha\theta = \int_{l_a}^{l_b} q^{\rho}(\theta)\Diamond_\alpha\theta = 1$$

where

$$M_1 = \sup_{l_a < \tau \leq \rho(l_b)} \left|\pi^{\nabla}(\tau)\right|, M_2 = \sup_{\sigma(l_a) \leq \tau < l_b} \left|\pi^{\Delta}(\tau)\right|$$

and

$$M = \max\{M_1, M_2\}$$

Proof. Consider the left hand side of (3.5.1), which can be written as

$$|\pi(t)\left(\alpha\int_{l_a}^{l_b} q^{\sigma}(\theta)\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}(\theta)\Diamond_{\alpha}\theta\right)$$

$$-\left(\alpha\int_{l_a}^{l_b} \pi^{\sigma}(\theta)q^{\sigma}(\theta)\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}(\theta)\pi\right.$$

$$\leq \alpha\int_{l_a}^{l_b} q^{\sigma}(\theta)|\pi(t)-\pi^{\sigma}(\theta)|\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}^{l_b}$$

$$\leq \alpha\int_{l_a}^{l_b} q^{\sigma}(\theta)\left|\int_{\sigma(\theta)}^{t} \pi(\tau)^{\Delta}\Delta\tau\right|\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}$$

$$\leq M_2\alpha\int_{l_a}^{l_b} q^{\sigma}|\sigma(\theta)-t|\Diamond_{\alpha}\theta + (1-\alpha)M_1\int_{l_a}^{l_b} q$$

$$\leq M\left(\alpha\int_{l_a}^{l_b} q^{\sigma}|\sigma(\theta)-t|\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}|\rho\right.$$

Remark 3.22. When $\alpha = 1$, we get

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\sigma}(\theta)\pi^{\sigma}(\theta)\Delta\theta\right| \leq \int_{l_a}^{l_b} q^{\sigma}(\theta)|\sigma(s)-t|l$$

which is the Weighted Ostrowski inequality developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\right|$$

$$\leq \int_{l_a}^{l_b} q^{\rho}(\theta)|\rho(s)-t|M\nabla\theta$$

which is Ostrowski inequality developed using nabla calculus (see (3.2.4)).

Corollary 3.23. Assume $l_a, l_b, \theta, t, \tau \in \mathbb{T}$ with $l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$. Let $\pi^{\nabla_{\alpha}}$ exist and corresponding $\pi^{\Delta}, \pi^{\nabla}$ are continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then the following inequality

$$\left|\pi(t) - \left(\alpha\int_{l_a}^{l_b} \pi^{\sigma}(\theta)q^{\sigma}(\theta)\Diamond_{\alpha}\theta + (1-\alpha)\int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\Diamond_{\alpha}\theta\right)\right|$$

$$\leq M\left[\alpha\left(\int_{l_a}^{l_b} |\sigma(\theta)-t|^{p}\Diamond_{\alpha}\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} \left(q^{\sigma}(\theta)\right)^{q}\Diamond_{\alpha}\theta\right)^{\frac{1}{q}}\right.$$

$$\left.+ (1-\alpha)\left(\int_{l_a}^{l_b} |\rho(\theta)-t|^{p}\Diamond_{\alpha}\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} |q^{\rho}(\theta)|^{q}\Diamond_{\alpha}\theta s\right)^{\frac{1}{q}}\right]$$

will hold, where M, M_1, M_2 and the weight function q are same as in Theorem 3.21

Proof. Since $q^{\sigma}, q^{\rho} \geq 0$, so the right hand side of (3.5.1) can be written as

$$M\left(\alpha\int_{l_a}^{l_b} |q^{\sigma}(\theta)||\sigma(\theta)-t|\Diamond_{\alpha}\theta\right.$$

$$+ (1-\alpha)\int_{l_a}^{l_b} |q^{\rho}(\theta)||\rho(\theta)$$

$$\left.- t|\Diamond_{\alpha}\theta\right)$$

now by using Holder's inequality (Theorem 2.10), we can get the desired result. Where $\frac{1}{p}+\frac{1}{q}=1$, and $p > 1$.

Remark 3.24. When $\alpha = 1$, we get

$$\left|\pi(t) - \int_{l_a}^{l_b} \pi^{\sigma}(\theta)q^{\sigma}(\theta)\Delta\theta\right|$$

$$\leq \left[\left(\int_{l_a}^{l_b} |t-\sigma(\theta)|^{p}\Delta\theta\right)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} \left(q^{\sigma}(\theta)\right)^{q}\Delta\theta\right)^{\frac{1}{q}}\right]M$$

which is the Weighted Ostrowski inequality developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\Big)|$$

$$\leq M\left(\int_{l_a}^{l_b} |\rho(\theta)\right.$$

$$\left.- t|^{p}\nabla\theta)^{\frac{1}{p}}\left(\int_{l_a}^{l_b} |q^{\rho}(\theta)|^{q}\nabla\theta s\right)^{\frac{1}{q}}\right],$$

which is Weighted Ostrowski inequality developed using nabla calculus (see (3.2.8)).

Corollary 3.25. Assume $l_a, l_b, \theta, t, \tau \in \mathbb{T}$ with $l_a <$

l_b and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$. Let π^{∇_α} exist and corresponding $\pi^{\Delta}, \pi^{\nabla}$ are continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then the following inequality

$$\Bigg|\pi(t) - \Bigg(\alpha \int_{l_a}^{l_b} \pi^{\sigma}(\theta) q^{\sigma}(\theta) \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q^{\rho}(\theta) \pi^{\rho}(\theta) \Diamond_\alpha \theta\Bigg)\Bigg|$$

$$\leq MM_3[\tilde{g}_2(l_a, \theta) + \tilde{g}_2(l_b, \theta) + \alpha(1-\alpha) H(\theta, t)]$$

will hold. Where

$$H(\theta, t) = \int_{l_a}^{l_b} |\sigma(\theta) - t| \nabla\theta + \int_{l_a}^{l_b} |\rho(\theta) - t| \Delta\theta$$

$$M_3 = \sup_{l_a \leq \theta < l_b} q(\theta)$$

M, M_1, M_2 and the weight function q are same as in Theorem 3.21.

Proof. We have

$$\pi(t) - \Bigg(\alpha \int_{l_a}^{l_b} \pi^{\sigma}(\theta) q^{\sigma}(\theta) \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q^{\rho}($$

$$\leq M \Bigg(\alpha \int_{l_a}^{l_b} q^{\sigma}(\theta) |\sigma(\theta) - t| \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q$$

$$\leq M \Bigg[\alpha M_3 \Bigg(\int_{l_a}^{t} \big(t - \sigma(\theta)\big) \Diamond_\alpha \theta + \int_{t}^{l_b} (\sigma(\theta) - t)$$

$$+ (1-\alpha) M_3 \Bigg(\int_{l_a}^{t} \big(t - \rho(\theta)\big) \Diamond_\alpha \theta + \int_{t}^{l_b} (\rho(\theta) -$$

$$= MM_3 \Bigg[\alpha \Bigg(\alpha \Bigg(\int_{l_a}^{t} \big(t - \sigma(\theta)\big) \Delta\theta + \int_{t}^{l_b} (\sigma(\theta) -$$

$$+ \int_{t}^{l_b} (\sigma(\theta) - t) \nabla\theta\Bigg)\Bigg) + (1-\alpha) \Bigg(\alpha \Bigg(\int_{l_a}^{t} (t - \rho$$

$$+ (1-\alpha) \Bigg(\int_{l_a}^{t} \big(t - \rho(\theta)\big) \nabla\theta + \int_{t}^{l_b} (\rho(\theta) - t) \nabla$$

$$\leq MM_3[\alpha[g_2(l_a, t) + g_2(l_b, t)] + (1-\alpha)[\hat{g}_2(l_a,$$

$$+ \alpha(1-\alpha) \Bigg\{\Bigg(\int_{l_a}^{t} \big(t - \sigma(\theta)\big) \nabla\theta + \int_{t}^{l_b} (\sigma(\theta) - t$$

$$= MM_3[\alpha[g_2(l_a, t) + g_2(l_b, t)] + (1-\alpha)[\hat{g}_2(l_a,$$

$$+ \alpha(1-\alpha) \Bigg(\int_{l_a}^{l_b} |\sigma(\theta) - t| \nabla\theta + \int_{l_a}^{l_b} |\rho(\theta) - t|$$

By applying the definition 2.11, we get the desired result.

Remark 3.26. When $\alpha = 1$, we get

$$\Bigg|\pi(t) - \int_{l_a}^{l_b} q^{\sigma}(\theta) \pi^{\sigma}(\theta) \Delta\theta\Bigg|$$

$$\leq MM_3[g_2(l_a, t) + g_2(l_b, t)]$$

which is the Weighted Ostrowski inequality developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta) \pi^{\rho}(\theta) \nabla\theta\Bigg)|$$

$$\leq MM_3[\hat{g}_2(l_a, t) + \hat{g}_2(l_b, t)]$$

which is Weighted Ostrowski inequality developed using nabla calculus (see (3.2.11)).

Corollary 3.27. Assume $l_a, l_b, \theta, t, \tau \in \mathbb{T}$ with $l_a < l_b$ and $\pi: [l_a, l_b]_{\mathbb{T}} \to \mathbb{R}$. Let π^{∇_α} exist and corresponding $\pi^{\Delta}, \pi^{\nabla}$ are continuous on $\mathbb{T}^{\omega}$ and $\mathbb{T}_{\omega}$ respectively. Then the following inequality

$$\Bigg|\pi(t) - \Bigg(\alpha \int_{l_a}^{l_b} q^{\sigma}(\theta) \pi^{\sigma}(\theta) \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q^{\rho}(\theta) \pi^{\rho}(\theta) \Diamond_\alpha \theta$$

$$\leq M \Bigg[\alpha \Bigg(\frac{l_b - \sigma(a)}{2} + \Bigg|t - \frac{l_b + \sigma(a)}{2}\Bigg|\Bigg) + (1-\alpha) \Bigg(\frac{\rho(l_b) - l_a}{2} +$$

will hold, where M, M_1, M_2 and the weight function q are same as in Theorem 3.21.

Proof. We have

$$\Bigg|\pi(t) - \Bigg(\alpha \int_{l_a}^{l_b} q^{\sigma}(\theta) \pi^{\sigma}(\theta) \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q^{\rho}(\theta) \pi^{\rho}(\theta) \Diamond_\alpha \theta$$

$$\leq \Bigg(\alpha \int_{l_a}^{l_b} q^{\sigma}(\theta) |\sigma(\theta) - t| \Diamond_\alpha \theta + (1-\alpha) \int_{l_a}^{l_b} q^{\rho}(\theta) |\rho(\theta) - t| \Diamond$$

$$\leq M \Bigg(\alpha \sup_{l_a \leq \theta < l_b} \{|\sigma(\theta) - t|\} + (1-\alpha) \sup_{l_a < \theta \leq l_b} \{|\rho(\theta) - t|\}\Bigg)$$

$$\leq M \Bigg[\alpha \Bigg(\frac{l_b - \sigma(a)}{2} + \Bigg|t - \frac{l_b + \sigma(a)}{2}\Bigg|\Bigg) + (1-\alpha) \Bigg(\frac{\rho(l_b) - l_a}{2} +$$

Remark 3.28. When $\alpha = 1$, we get

$$\left|\pi(t) - \int_{l_a}^{l_b} q^{\sigma}(\theta)\pi^{\sigma}(\theta)\Delta\theta\right| \leq M\left(\frac{l_b - \sigma(a)}{2} + \left|t - \frac{l_b + \sigma(a)}{2}\right|\right)$$

which is the Weighted Ostrowski inequality developed using delta calculus (see [3]).

When $\alpha = 0$, we get

$$|\pi(t) - \int_{l_a}^{l_b} q^{\rho}(\theta)\pi^{\rho}(\theta)\nabla\theta\Big)| \leq M\left(\frac{\rho(l_b) - l_a}{2} + \left|-t + \frac{l_a + \rho(l_b)}{2}\right|\right)$$

which is Weighted Ostrowski inequality developed using nabla calculus [see(3.2.13)].

Conclusion

In this paper, we have introduced and rigorously developed Diamond- α Ostrowski-type inequalities within the framework of time scales, leveraging the unified approach of delta and nabla calculus. The results extend classical inequalities by incorporating the dynamic nature of time scales, thus bridging discrete and continuous cases. Specifically, the generalization of the Montgomery identity (Theorem 3.17, Eq. (3.3.1)) through Diamond- α calculus has allowed for the derivation of both standard and weighted Ostrowski inequalities (Theorems 3.19 and 3.21, Eqs. (3.4.1) and (3.5.1)), presenting these as specialized forms for specific time scales, such as $T = \mathbb{R}$ and $T = \mathbb{Z}$.

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