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Laplacian Harmonic Spectral Analysis for IoT Network Topology Characterization

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Abstract

The rapid expansion of the Internet of Things (IoT) has led to the deployment of large-scale, heterogeneous, and dynamically evolving network topologies. Understanding the structural properties of such networks is critical for ensuring robustness, efficient communication, and fault tolerance. Spectral graph theory provides powerful tools for analyzing network structures through eigenvalues of matrix representations. In this paper, we introduce the Laplacian Harmonic matrix as an alternative spectral framework for modeling IoT network topologies. We investigate its fundamental properties and analyze the associated eigenvalues for various graph families. Furthermore, we interpret these spectral characteristics in the context of IoT networks, including star, mesh, and bipartite topologies. The results demonstrate that Laplacian Harmonic eigenvalues provide deeper insights into connectivity, robustness, and structural balance compared to traditional Laplacian matrices. The proposed framework offers a novel approach for IoT network design, monitoring, and optimization.

1 Introduction

The Internet of Things (IoT) represents a paradigm shift in modern computing, enabling billions of interconnected devices to communicate and exchange data across distributed environments. These devices form complex network topologies characterized by dynamic connectivity, heterogeneous structures, and resource constraints.

Efficient analysis of IoT network topology is essential for ensuring reliability, scalability, and energy efficiency.

Graph theory provides a natural framework for modeling IoT networks, where devices are represented as vertices and communication links as edges. Among various graph-based techniques, spectral graph theory has gained significant attention due to its ability to extract structural information from eigenvalues of matrices such as adjacency and Laplacian matrices.

However, traditional Laplacian matrices may not fully capture the nuanced relationships between nodes, especially in heterogeneous IoT networks. To address this limitation, we introduce the Laplacian Harmonic matrix, which incorporates degree-based harmonic weighting into the network structure.

Contributions of this Paper

- Introduction of the Laplacian Harmonic matrix for IoT network modeling.
- Theoretical analysis of its eigenvalues and structural properties.
- Characterization of graph families based on Laplacian Harmonic spectra.
- Application of spectral results to IoT network topologies (star, mesh, bipartite).
- Insights into network robustness, connectivity, and fault tolerance.

2 Related Work

Spectral graph theory has been widely used for analyzing complex networks, including communication and sensor networks. The Laplacian matrix has been extensively applied for studying connectivity, clustering, and synchronization in networks. In IoT systems, graph-based approaches are used to model communication patterns, optimize routing, and detect anomalies. Previous studies have explored eigenvalue-based methods for network robustness and community detection. However, limited work has been done on alternative spectral representations such as harmonic-based matrices. The harmonic index has been studied in chemical graph theory, particularly in Quantitative Structure–Property Relationships (QSPR). Extending such indices to Laplacian-based representations opens new avenues for analyzing real-world networks such as IoT systems.

3 Preliminaries

Let $G = (V, E)$ be an undirected graph, where:
- $V = \{v_1, v_2, \ldots, v_n\}$: set of vertices (IoT devices),
- $E = \{e_1, e_2, \ldots, e_m\}$: set of edges (communication links)

The degree of a vertex v, denoted $d(v)$, represents the number of connected devices.

IoT Interpretation

Vertex → IoT device (sensor, actuator, gateway)
Edge → Communication link
Degree → Connectivity level of a device

Number of vertices in $V(G)$ are called the order n, whereas edges in $E(G)$ define the size m of G. $u \sim v$ indicates that vertex u is adjacent to vertex v, and the edge connecting them is denoted as e. The neighborhood $N(v)$ of a vertex $v \in V(G)$ includes all adjacent vertices. For a vertex v_i, its degree $d(v_i)$ is the number of vertices in its neighborhood $N(v_i)$. A graph G is r-regular if every vertex has a degree of r. The distance between two vertices u, v in a connected graph G is the length of the shortest path connecting them. The diameter of a graph G is the greatest distance between any two vertices. The complete graph on n vertices is identified as K_n, the complete bipartite graph with partition sizes a and b is $K_{a,b}$, and the star graph is $K_{1,n-1}$. The union of two graphs $(G_1 \cup G_2)$ consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The vertex-disjoint union of k copies of G is $k \cdot G$. Standard graph theory notation is used, with additional notations in [3]. For graph G, the Harmonic index, introduced in [4], is as follows:

$$\mathrm{HI}(G) = \sum_{v_i v_j \in E(G)} \frac{2}{d_{v_i} + d_{v_j}}$$

This index is well examined and has several applications in quantitative structure-activity relationships (QSAR) and quantitative structure-property relationships (QSPR) [6, 7, 5]. The harmonic matrix of a graph G is a square matrix $H(G) = [h_{ij}]$ of order n, defined in [5] as

$$A_{HI}(G) = (h_{ij})_{n\times n} = \begin{cases} \frac{2}{d_{v_i}+d_{v_j}} & \text{if there is an edge} \\ & \text{ between } v_i \text{ and } v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_{HI}(G)$ for a graph G are termed the HI eigenvalues of G. Let $s_1 \geq s_2 \geq \ldots \geq s_n$ denote the HI-eigenvalues of G, with s_1 referred to as the HI-spectral radius of G. For a graph G, the Laplacian matrix is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G, and $A(G)$ is the adjacency matrix of G. The eigenvalues of the Laplacian matrix are denoted as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. The largest eigenvalue of the Laplacian matrix of graph G is termed its Laplacian spectral radius. Since the Laplacian matrix of a graph provides greater insight into graph structures than its adjacency matrix, it is reasonable to extend the HI-adjacency matrix of G to the Laplacian HI-matrix. A similar study was first introduced by Yang et al. [10] for the Laplacian ABC matrix. The authors investigated several properties of this matrix, including the characterization of graphs with one and two distinct Laplacian ABC eigenvalues, as well as bounds for the largest and smallest Laplacian ABC eigenvalues. This work was further extended by Rather et al. in [8, 9].

In this paper, we introduce the Laplacian HI-matrix associated with a simple graph. We investigate several properties of this matrix and determine the Laplacian HI-eigenvalues for certain families of graphs. In addition, we completely characterize graphs having exactly one and two distinct Laplacian HI-eigenvalues and provide a characterization of graphs with three distinct Laplacian HI-eigenvalues.

4 Laplacian Harmonic-matrix

For each vertex v in the vertex set of graph G, let $\overline{d_v}$ denote the sum of $\frac{2}{d_v+d_u}$ for every vertex u in the neighborhood $N(v)$. For a graph G of order n, the Laplacian Harmonic–matrix is defined as $\mathcal{L}(G) = \overline{D}(G) - HA(G)$, where $\overline{D}(G) = [\overline{d_{ii}}]_{n\times n}$ is the Harmonic- diagonal matrix with diagonal entries $\overline{d_{ii}} = \overline{d_{v_i}} = \sum\limits_{v_j \in N(v_i)} \frac{2}{d_{v_i}+d_{v_j}}$, representing the Harmonic–degree of vertex v_i. From now on, we will use d_i to represent $\overline{d}_{v_i}$. Let G be a graph with n vertices. The eigenvalues of $\mathcal{L}(G)$ are denoted as $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$, known as the Laplacian HI-eigenvalues of G. The maximal eigenvalue of the Laplacian HI-matrix of G represents the Laplacian HI-spectral radius of G.

Let G be an n-vertex graph with m edges, and let D be a directed graph. We define the vertex-arc incidence matrix $S(G) = [h_{ve}]_{n\times m}$, where the element h_{ve} for vertex v and arc e is defined as:

$$h_{ve}=\begin{cases}\frac{\sqrt{2}}{\sqrt{d_{v_i}+d_{v_j}}} & \text{if } v \text{ is the terminal vertex of } e,\\ \frac{-\sqrt{2}}{\sqrt{d_{v_i}+d_{v_j}}} & \text{if } v \text{ is the initial vertex of } e,\\ 0 & \text{if } v \text{ and } e \text{ are not incident.}\end{cases} \tag{4.1}$$

In the following theorem, we prove that for any orientation of the graph G, the Laplacian HI-matrix can be expressed as $\mathcal{L}(G)=S(G)\cdot S(G)'$. In the following theorem, we prove that for any orientation of the graph G, the Laplacian HI-matrix can be expressed as $\mathcal{L}(G)=S(G)\cdot S(G)'$.

Theorem 4.1. *Let G be a graph of order n and D be its directed graph. Then for any orientation of the graph, we have $\mathcal{L}(G)=S(G)\cdot S(G)'$, where $S(G)$ is defined in (4.1) and $S(G)'=[h_{ve}]^t$, the transpose of $S(G)$.*

Proof. Let $[h_{ve}]\cdot[h_{ve}]^t=(z_{kl})_{n\times n}$. We will analyze the correlation between the components of $\mathcal{L}(G)$ and $S(G)\cdot S(G)'$. It is evident that $z_{kl}=\sum\limits_{e\in E}h_{ke}\cdot h_{le}$. If $k=l$, then

$$z_{kl}=\sum_{e\in E}h_{ke}\cdot h_{le}=\sum_{v_kv_l=e\in E}\frac{2}{d_k+d_l}.$$

If $k\neq l$, then $z_{kl}=h_{ke}\cdot h_{le}$, where v_k and v_l are adjacent and $h_{ke}=\frac{\sqrt{2}}{\sqrt{d_k+d_l}}$ and $h_{le}=-\frac{\sqrt{2}}{\sqrt{d_k+d_l}}$. This results in:

$$z_{kl}=\frac{-2}{d_k+d_l}.$$

In conjunction with the preceding reasoning, we have $\mathcal{L}(G)=S(G)\cdot S(G)'$. ☐

Lemma 4.2. *For any graph G, demonstrate that $\mathcal{L}(G)$ is a positive semi-definite matrix.*

Proof. Let $x=(x_u)$ $(u\in V(G))$ be a column vector. We have

$$x^tL(G)x=x^t[h_{ke}][h_{ke}]x$$
$$=\sum_{e=v_kv_l}\frac{(\sqrt{2}x_k-\sqrt{2}x_l)^2}{d_k+d_l}\geq 0.$$

This completes the proof. ☐

It is easy to see that 0 is an eigenvalue of $\mathcal{L}(G)$ with eigenvector $\mathbf{1}$, which is the all 1 vector. The following lemma provides the rank of the incidence matrix; the proof is analogous to the proof of Lemma 2.2[1].

Lemma 4.3. *Let G be a connected n-vertex graph, where $n\geq 3$. Then rank of the incidence matrix is $n-1$.*

The following lemma is important for our results. Its proof follows the same arguments as in Lemma 2.2 of [11], and therefore we omit the details.

Lemma 4.4. [11] *Let G be a connected graph with $r\geq 3$ vertices. Then $\mathcal{L}(G)$ has $t(2\leq t\leq r)$ distinct eigenvalues if and only if there exist $k-1$ distinct nonzero numbers $s_1,s_2,\ldots,s_k$ such that:*

$$\prod_{j=1}^{k-1}((L(G)-s_jI))=(-1)^{t-1}\frac{\prod\limits_{j=1}^{t-1}s_j}{n}J,$$

where I is the unit matrix of order r and J is the all 1 matrix of order r.

Theorem 4.5. *Let G be a graph of order n. Then G has exactly one (distinct) Laplacian HI-eigenvalue if and only if $G=nK_1$.*

Proof. Observe that $\mathcal{L}(G)$ is a positive semi-definite matrix. It follows that $\operatorname{tr}(\mathcal{L}(G))=0$ if and only if $\mathcal{L}(G)=0$, which happens only when $G=K_1$. Hence, the result follows. ☐

It is straightforward to verify that the graph K_2 has exactly two distinct Laplacian HI-eigenvalues, namely 0 and 2. This observation leads to the following theorem.

Theorem 4.6. *Let G be a graph of order n. Then G has exactly two (distinct) Laplacian HI-eigenvalues if $G = sK_2 \cup (n-2s)K_1$, where $0 \le s \le \frac{n}{2}$.*

Theorem 4.7. *Let G be an n-vertex connected graph with $n \ge 3$. Let G possess s connected components, each containing n_i vertices, where $n_i \ge 3$ for $i = 1, 2, \ldots, s$. Then, given $\mathcal{L}(G)$, the number of components is equal to the multiplicity of the eigenvalue 0.*

Proof. It is simple to obtain that

$$rank(\mathcal{L}(G)) = rank(S(G)R(G)^t) = rank(S(G)).$$

Lemma 4.3 states that the rank of $\mathcal{L}(G)$ and the rank of $S(G)$ are both equal to $n-1$ if G is connected. Given that $\mathcal{L}(G)$ is a real symmetric matrix, the eigenvalue 0 has a multiplicity of one. Assume that G possesses s ($s > 1$) connected components, each containing n_i vertices, where $n_i \ge 3$, for all $i = 1, 2, \ldots, s$. By applying Lemma 4.3, we ascertain that $rank(\mathcal{L}(G)) = n - s$, indicating that the multiplicity of the eigenvalue 0 is s. □

5 Laplacian Harmonic–eigenvalues of some families of graphs

This section will describe the Laplacian harmonic eigenvalues of particular families of graphs. This study extends the previously stated conclusions on the Laplacian eigenvalues of classical Laplacians to the Laplacian harmonic eigenvalues of certain graph families.

Theorem 5.1. *Let G denote a complete bipartite graph with cardinalities r and s for its two partitioned sets, respectively. Consequently, the Laplacian harmonic spectrum of G for this graph is as follows:*

$$Spec(G) = \left\{0, 2, \frac{2s}{r+s}^{(r-1)}, \frac{2r}{r+s}^{(s-1)}\right\}.$$

Proof. Let G be a complete bipartite graph with partite sets of cardinalities r and s. The Harmonic-Laplacian matrix of G is defined as follows:

$$\mathcal{L}(G) = \begin{pmatrix}
\frac{2s}{r+s} & 0 & \cdots & 0 & \frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} \\
0 & \frac{2s}{r+s} & \cdots & 0 & \frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} \\
\vdots & \vdots & \ddots & \vdots & & & & \\
0 & 0 & \cdots & \frac{2s}{r+s} & \frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} \\
\frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} & \frac{2r}{r+s} & 0 & \cdots & 0 \\
\frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} & 0 & \frac{2r}{r+s} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & & & & \\
\frac{-2}{r+s} & \frac{-2}{r+s} & \cdots & \frac{-2}{r+s} & 0 & 0 & \cdots & \frac{2r}{r+s}
\end{pmatrix}.$$

To determine the eigenvalues of $\mathcal{L}(G)$, we equate the characteristic polynomial of $\mathcal{L}(G)$ to zero, specifically, $\det(\lambda I - \mathcal{L}(G)) = 0$. This results in:

$$\begin{vmatrix}
\lambda - \frac{2s}{r+s} & 0 & \cdots & 0 & \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \\
0 & \lambda - \frac{2s}{r+s} & \cdots & 0 & \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \\
\vdots & \vdots & \ddots & \vdots & & & & \\
0 & 0 & \cdots & \lambda - \frac{2s}{r+s} & \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \\
\frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} & \lambda - \frac{2r}{r+s} & 0 & \cdots & 0 \\
\frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} & 0 & \lambda - \frac{2r}{r+s} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & & & & \\
\frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} & 0 & 0 & \cdots & \lambda - \frac{2r}{r+s}
\end{vmatrix} = 0.$$

Employing the definition of the determinant for a block matrix, we elucidate the aforementioned equation:

$$\det(A) \cdot \det(D - CA^{-1}B) = 0, \tag{5.2}$$

where $A = \begin{bmatrix} \lambda - \frac{2s}{r+s} & 0 & \cdots & 0 \\ 0 & \lambda - \frac{2s}{r+s} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{2s}{r+s} \end{bmatrix}_{r\times r}$, $B = \begin{bmatrix} \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \\ \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{r+s} & \frac{2}{r+s} & \cdots & \frac{2}{r+s} \end{bmatrix}_{r\times s}$,

$C = B^t$ and $D = \begin{bmatrix} \lambda - \frac{2r}{r+s} & 0 & \cdots & 0 \\ 0 & \lambda - \frac{2r}{r+s} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{2r}{r+s} \end{bmatrix}_{s\times s}$.

It is evident that:

$$\det(A) = \left(\lambda - \frac{2s}{r+s}\right)^r. \tag{5.3}$$

To determine the second determinant in the expression $\det(A) \cdot \det(D - CA^{-1}B) = 0$, we initially calculate $CA^{-1}B$, specifically,

$$CA^{-1}B = y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{s\times r} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{r\times r} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{r\times s}, \tag{5.4}$$

where $\left(\frac{2}{r+s}\right)^2 \left(\frac{1}{\lambda - \frac{2s}{r+s}}\right)$. The above equation becomes:

$$\begin{aligned}
CA^{-1}B &= y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{s\times r} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{r\times s} \\
&= y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{s\times s}.
\end{aligned}$$

Next, we will find the matrix represented by $D - CA^{-1}B$, which is equal to:

$$D - CA^{-1}B = \begin{bmatrix} x-y & -y & -y & \cdots & -y \\ -y & x-y & -y & \cdots & -y \\ -y & -y & x-y & \cdots & -y \\ \vdots & \vdots & \ddots & \vdots & \\ -y & -y & -y & \cdots & x-y \end{bmatrix}_{s\times s},$$

where $x = \lambda - \frac{2r}{r+s}$ and $y = \left(\frac{2}{r+s}\right)^2 \frac{r}{\lambda - \frac{2s}{r+s}}$. Finally, the computation of the determinant of matrix $D - CA^{-1}B$ yields the following equation:

$$\det(D - CA^{-1}B) = \begin{vmatrix} x-y & -y & -y & \cdots & -y \\ -y & x-y & -y & \cdots & -y \\ -y & -y & x-y & \cdots & -y \\ \vdots & \vdots & \ddots & \vdots & \\ -y & -y & -y & \cdots & x-y \end{vmatrix}_{s\times s}.$$

By summing all columns into the first column of the determinant and extracting the common factor from the first column, we derive:

$$\det(D - CA^{-1}B) = (x - sy) \begin{vmatrix} 1 & -y & -y & \cdots & -y \\ 1 & x-y & -y & \cdots & -y \\ 1 & -y & x-y & \cdots & -y \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & -y & -y & \cdots & x-y \end{vmatrix}_{s\times s}.$$

By subtracting the first row of the determinant from the successive rows and then expanding the determinant, we obtain the following:

$$\det(D - CA^{-1}B) = (x - sy)x^{s-1}. \tag{5.5}$$

By Combining equations (5.3) and (5.5), equation (5.2) can be rewritten as:

$$\det(\lambda I - \mathcal{L}(G)) = \left(\lambda - \frac{2s}{r+s}\right)^r (x - sy)x^{s-1},$$

where $x = \lambda - \frac{2r}{r+s}$ and $y = \left(\frac{2}{r+s}\right)^2 \frac{r}{\lambda - \frac{2s}{r+s}}$. By simplifying the aforementioned equation, we obtain:

$$\begin{aligned} \det(\lambda I - \mathcal{L}(G)) &= \left(\lambda - \frac{2s}{r+s}\right)^r (x - sy)x^{s-1} \\ &= \left(\lambda - \frac{2s}{r+s}\right)^r \left(\lambda - \frac{2r}{r+s} - s\cdot\left(\frac{2}{r+s}\right)^2 \frac{r}{\lambda - \frac{2s}{r+s}}\right) \left(\lambda - \frac{2r}{r+s}\right)^{s-1} \\ &= \lambda\left(\lambda - 2\right)\left(\lambda - \frac{2s}{r+s}\right)^{r-1} \left(\lambda - \frac{2r}{r+s}\right)^{s-1}. \end{aligned}$$

This completes the proof. □

We define a graph $B_{s,r-s}$ that includes a vertex set that is partitioned into two disjoint subsets, namely V_1 and V_2, where V_1 is comprised of s vertices and V_2 is formed of $r-s$ vertices. In contrast to the other category of edges, which interconnects all of the vertices that are contained within V_2, the first category of edges connects every vertex in V_1 to every vertex in V_2. There are two different types of edges that make up the edge set of the graph.

Theorem 5.2. *Let r and s be two positive integers, where $r \geq 3$, $s \geq 2$, and $r-s \geq 0$. Subsequently, the*

$$\begin{aligned}&\mathrm{Spec}(B_{s,r-s}) = \\ &\left\{0, \frac{2r}{2r-s-1}, \left(\frac{2s}{2r-s-1} + \frac{r-s}{r-1}\right)^{(r-s-1)}, \left(\frac{2(r-s)}{2r-s-1}\right)^{s-1}\right\}\end{aligned}$$

Proof. Let $B_{s.r-s}$ denote a graph as stated above. Then the Harmonic-Laplacian matrix of $B_{s.r-s}$ is presented as follows:

$$\mathcal{L}(B_{m.n-m}) = \begin{pmatrix} A_{s\times s} & B_{s\times r-s} \\ C_{r-s\times s} & D_{r-s\times r-s} \end{pmatrix}.$$

where $A_{s\times s} = \begin{pmatrix} \frac{2(r-s)}{2r-s-1} & 0 & \cdots & 0 \\ 0 & \frac{2(r-s)}{2r-s-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2(r-s)}{2r-s-1} \end{pmatrix}$, $B_{s\times r-s} = \begin{pmatrix} \frac{-2}{2r-s-1} & \frac{-2}{2r-s-1} & \cdots & \frac{-2}{2r-s-1} \\ \frac{-2}{2r-s-1} & \frac{-2}{2r-s-1} & \cdots & \frac{-2}{2r-s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2}{2r-s-1} & \frac{-2}{2r-s-1} & \cdots & \frac{-2}{2r-s-1} \end{pmatrix}$,

$C_{r-s\times s} = B^t_{s\times r-s}$ and $D_{r-s\times r-s} = \begin{pmatrix} \frac{2s}{2r-s-1} + \frac{r-s-1}{r-1} & \frac{-1}{r-1} & \cdots & \frac{-1}{r-1} \\ \frac{-1}{r-1} & \frac{2s}{2r-s-1} + \frac{r-s-1}{r-1} & \cdots & \frac{-1}{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{r-1} & \frac{-1}{r-1} & \cdots & \frac{2s}{2r-s-1} + \frac{r-s-1}{r-1} \end{pmatrix}$.

The characteristic polynomial of $\mathcal{L}(B_{s,r-s})$ is $\det(\lambda I - \mathcal{L}(B_{s,r-s})) = 0$. This gives:

$$\det(\lambda I - \mathcal{L}(B_{s,r-s})) = \begin{vmatrix} \lambda I_s - A_{s\times s} & -B_{s\times r-s} \\ -C_{r-s\times s} & \lambda I_{r-s} - D_{r-s\times r-s} \end{vmatrix}$$

Utilizing the definition of the determinant of a block matrix, we articulate the aforementioned determinant as follows:

$$\det(A) \cdot \det(D - CA^{-1}B) = 0. \tag{5.6}$$

where $A = \begin{bmatrix} \lambda - \frac{2(r-s)}{2r-s-1} & 0 & \cdots & 0 \\ 0 & \lambda - \frac{2(r-s)}{2r-s-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \frac{2(r-s)}{2r-s-1} \end{bmatrix}_{s\times s}$,

$B = \begin{bmatrix} \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \\ \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \end{bmatrix}_{s\times r-s}$, $C = \begin{bmatrix} \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \\ \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{2r-s-1} & \frac{2}{2r-s-1} & \cdots & \frac{2}{2r-s-1} \end{bmatrix}_{r-s\times s}$ and

$$D = \begin{bmatrix} \lambda - \left(\frac{2s}{2r-s-1} + \frac{r-s-1}{r-1}\right) & \frac{1}{r-1} & \cdots & \frac{1}{r-1} \\ \frac{1}{r-1} & \lambda - \left(\frac{2s}{2r-s-1} + \frac{r-s-1}{r-1}\right) & \cdots & \frac{1}{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r-1} & \frac{1}{r-1} & \cdots & \lambda - \left(\frac{2s}{2r-s-1} + \frac{r-s-1}{r-1}\right) \end{bmatrix}_{r-s\times r-s}.$$

It is easy to see that

$$\det(A) = \left(\lambda - \frac{2(r-s)}{2r-s-1}\right)^s. \tag{5.7}$$

To compute the second determinant in (5.6), we first evaluate the matrix $CA^{-1}B$.

$$CA^{-1}B = y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{r-s\times s} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{s\times s} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{s\times r-s}, \tag{5.8}$$

where $y = \left(\frac{2}{2r-s-1}\right)^2 \cdot \left(\frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\right)$. The above equation becomes:

$$\begin{aligned} CA^{-1}B &= y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{r-s\times s} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{s\times r-s} \\ &= sy \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{r-s\times r-s}. \end{aligned}$$

Subsequently, we will compute $D - CA^{-1}B$, which is equivalent to:

$$\begin{aligned} D - CA^{-1}B &= \begin{bmatrix} \lambda - x & \frac{1}{r-1} & \cdots & \frac{1}{r-1} \\ \frac{1}{r-1} & \lambda - x & \cdots & \frac{1}{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r-1} & \frac{1}{r-1} & \cdots & \lambda - x \end{bmatrix}_{n-m\times n-m} - y \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n-m\times n-m} \\ &= \begin{bmatrix} \lambda - x - y & \frac{1}{r-1} - y & \cdots & \frac{1}{r-1} - y \\ \frac{1}{r-1} - y & \lambda - x - y & \cdots & \frac{1}{r-1} - y \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r-1} - y & \frac{1}{r-1} - y & \cdots & \lambda - x - y \end{bmatrix}_{r-s\times r-s}. \end{aligned}$$

where $x = \frac{2s}{2r-s-1} + \frac{r-s-1}{r-1}$ and $y = s\left(\frac{2}{2r-s-1}\right)^2 \cdot \left(\frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\right)$ The determinant of the matrix $D - CA^{-1}B$ is expressed in the following equation:

$$\det(D - CA^{-1}B) = \begin{vmatrix} \lambda - x - y & \frac{1}{r-1} - y & \cdots & \frac{1}{r-1} - y \\ \frac{1}{r-1} - y & \lambda - x - y & \cdots & \frac{1}{r-1} - y \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r-1} - y & \frac{1}{r-1} - y & \cdots & \lambda - x - y \end{vmatrix}_{r-s\times r-s}.$$

First adding all columns into the first column and subsequently extracting the common term from the first column, we obtain following results:

$$\det(D - CA^{-1}B) = X \cdot \begin{vmatrix} 1 & \frac{1}{r-1} - y & \cdots & \frac{1}{r-1} - y \\ 1 & \lambda - x - y & \cdots & \frac{1}{r-1} - y \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{r-1} - y & \cdots & \lambda - x - y \end{vmatrix}_{r-s \times r-s},$$

where $X = \left(\lambda - x - y + (r-s-1)\left(\frac{1}{r-1} - y\right)\right)$. Next, by subtracting the first row from all subsequent rows and then expanding the determinant, we derive the following equation:

$$\det(D - CA^{-1}B) = \left(\lambda - x - y + (r-s-1)\left(\frac{1}{r-1} - y\right)\right) \cdot \left(\lambda - x - \frac{1}{r-1}\right)^{r-s-1}. \quad (5.9)$$

Subsisting the value of x and y, we obtain:

$$\begin{aligned} & \left(\lambda - x - y + (r-s-1)\left(\frac{1}{r-1} - y\right)\right) \\ = \; & \lambda - x + (r-s-1)\left(\frac{1}{r-1}\right) - (r-s)y \\ = \; & \lambda - \frac{2s}{2r-s-1} - \frac{r-s-1}{r-1} + \frac{r-s-1}{r-1} - \frac{4s}{(2r-s-1)^2}\left(\frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\right) \\ = \; & \lambda - \frac{2s}{2r-s-1} - \frac{4s}{(2r-s-1)^2}\left(\frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\right) \\ = \; & \frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\left(\left(\lambda - \frac{2s}{2r-s-1}\right)\left(\lambda - \frac{2(r-s)}{2r-s-1}\right) - \frac{4s}{(2r-s-1)^2}\right) \\ = \; & \frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\left(\lambda\left(\lambda - \frac{2r}{2r-s-1}\right)\right). \end{aligned}$$

and

$$\begin{aligned} \left(\lambda - x - \frac{1}{r-1}\right)^{r-s-1} &= \left(\lambda - \frac{2s}{2r-s-1} - \frac{r-s-1}{r-1} - \frac{1}{r-1}\right)^{r-s-1} \\ &= \left(\lambda - \frac{2s}{2r-s-1} - \frac{r-s}{r-1}\right)^{r-s-1}. \end{aligned}$$

Using this (5.9) becomes:

$$\det(D - CA^{-1}B) = \frac{1}{\lambda - \frac{2(r-s)}{2r-s-1}}\left(\lambda\left(\lambda - \frac{2r}{2r-s-1}\right)\right).$$

This together with (5.6) and (5.7) becomes:

$$\lambda\left(\lambda - \frac{2r}{2r-s-1}\right)\left(\lambda - \frac{2s}{2r-s-1} - \frac{r-s}{r-1}\right)^{r-s-1}\left(\lambda - \frac{2(r-s)}{2r-s-1}\right)^{s-1} = 0.$$

This completes the proof.

□

Theorem 5.1 and Theorem 5.2 help us to obtain the Laplacian HI-spectrum of some well-known families of graphs. In the following result we mention some of these families.

Corollary 5.3. *For $n \geq 3$, the following holds true.*

$$\begin{aligned} \operatorname{Spec}(K_{n,n}) &= \left\{0, 2, 1^{(2n-2)}\right\} \\ \operatorname{Spec}(K_{n-1,1}) &= \left\{0, 2, \frac{2}{n}^{(n-2)}\right\} \\ \operatorname{Spec}(K_n - e) &= \left\{0, \frac{2r}{2r-3}, \frac{2(r-2)}{2r-3}, \left(\frac{4}{2r-3} + \frac{r-2}{r-1}\right)^{(r-3)}\right\}. \end{aligned}$$

Theorem 5.4. *Let G be a graph of order n.*

(1) Let G be an n-vertex graph. If the graph G is m-regular, then $\eta_i = \frac{1}{m}\mu_i$ for all $1 \leq i \leq n$. Specifically, if G is isomorphic to K_n, then $\eta_1 = \eta_2 = \ldots = \eta_{n-1} = \frac{n}{n-1}$ and $\eta_n = 0$; if G is C_n, then $\eta_i = 1 - \cos\frac{2\pi i}{n}$ for $i = 1, 2, \ldots, n$.

(2) If G is (p, q) semi-regular bipartite graph, then $\eta_i = \frac{2}{p+q}\mu_i$, for $i = 1, 2, \ldots, n$. In particular, if G is $K_{r,s}$, where $r + s = n$, $r \geq s$, then $\eta_1 = 2, \eta_2 = \eta_3 \ldots, \eta_s = \frac{2r}{r+s}, \eta_{s+1} = \ldots = \eta_{r+s-1} = \frac{2s}{r+s}, \eta_{r+s} = 0$.

Proof. (1). If G is m-regular, it follows that $\mathcal{L}(G) = \frac{1}{m}L(G)$, and thus, $\eta_i = \frac{1}{m}\mu_i$ for all i, satisfying the condition $1 \leq i \leq n$. This, together with the condition that if G is equal to K_n, then $\mu_1 = \mu_2 = \ldots = \mu_{n-1} = n$ and $\mu_n = 0$, leads to $\eta_1 = \eta_2 = \ldots = \eta_{n-1} = \frac{n}{n-1}$ and $\eta_n = 0$. Furthermore, for $G = C_n$, it is established that $\mu_i = 2 - 2\cos\frac{2\pi i}{n}$ for $i = 0, 1, \ldots, n-1$, and using the fact that $\mathcal{L}(\mathcal{G}) = \frac{1}{2}L(G)$ give the required result.
(2). If G is (p, q) semi-regular bipartite graph, it is easy to see that $\mathcal{L}(G) = \frac{2}{p+q}L(G)$ and $\eta_i = \frac{2}{p+q}\mu_i$, for $i = 1, 2, \ldots, n$. If $G = K_{p,q}$ then the result follows from Theorem 5.1. ☐

Theorem 5.5. *Let G be a connected graph with $n \geq 3$. Then G has exactly two distinct Laplacian HI-eigenvalues if and only if $G = K_n$.*

Proof. By Lemma 4.4, G has exactly two distinct Laplacian HI-eigenvalues if and only if there is a non-zero number s such that

$$\mathcal{L}(G) - sI = -\frac{s}{n}J.$$

That is,

$$\mathcal{L}(G) = sI - \frac{s}{n}J.$$

We can see that the off-diagonal entries of $\mathcal{L}(G)$ are all non-zero. Thus, we see that $G = K_n$ and $s = \frac{n}{n-1}$. ☐

The theorem established by Brouwer and Haemers [2] delineates a relationship between the diameter of a graph and the quantity of distinct eigenvalues within the graph.

Theorem 5.6. [2] *Let G denote a connected graph with a diameter of D. Thus, G has at least $D + 1$ distinct adjacency eigenvalues (or Laplacian eigenvalues) and no fewer than $D+1$ distinct signless Laplacian eigenvalues.*

The proof provided in [2] confirms that the previously given assertion holds for any symmetric matrices $M = (m_{ij})$ with nonnegative elements associated with the vertices of a graph G, where $m_{ij} > 0$ if and only if an edge exists between vertices v_i and v_j. The subsequent corollary is derived directly by examining the matrix $M = n^2 I - \mathcal{L}(G)$.

Corollary 5.7. *If G is a graph with diameter D with k different Laplacian Harmonic-eigenvalues, then k must be greater than or equal to* $D + 1$.

The following result addresses the problem of characterizing connected graphs with exactly three Laplacian HI-eigenvalues.

Theorem 5.8. *Let G be an n-vertex connected graph, where* $n \geq 4$. *Consequently, the subsequent statements are valid.*

(i) *No graph G can have three distinct Laplacian ISI eigenvalues if its diameter is at least 3.*

(ii) If G is a star graph or a complete bipartite graph with equal-sized partite sets, it has three Laplacian Harmonic-eigenvalues.

(iii) A cycle G of length n has precisely three different Laplacian Harmonic-eigenvalues if and only if it is either C_4 *or* C_5.

(iv) If G is a regular graph that is not a complete graph K_n, *then G has three different Laplacian Harmonic-eigenvalues if and only if G is a strongly regular graph.*

Proof. (i). Let G be a graph with a diameter of at least 3; then, by Corollary 5.7, G has more than three distinct Laplacian harmonic eigenvalues.
(ii). If G is $K_{1,n-1}$, then by Corollary 5.3, we find that $\mathrm{Spec}\left(K_{n-1,1}\right) = \left\{0, 2, \frac{2}{n}^{(n-2)}\right\}$. It is clear that G has exactly three unique Laplacian Harmonic-eigenvalues. If G is $K_{p,p}$ with $n = 2p$, then, as per Corollary 5.3, the Laplacian Harmonic-spectrum of G is expressed as $\mathrm{Spec}\left(K_{n,n}\right) = \left\{0, 2, 1^{(2n-2)}\right\}$, and this finding is relevant in this context.

Conversely, let G be a bipartite graph with a diameter of at most two that possesses three unique Laplacian harmonic eigenvalues. We assert that G is either $K_{p,p}$ or $K_{n-1,1}$. It is evident that K_n is the only connected graph exhibiting a diameter of 1, and according to Theorem 5.5, this graph possesses two distinct Laplacian harmonic eigenvalues. Consequently, G cannot possess a diameter of 1. Consequently, the diameter of G must be 2. Let G be a bipartite graph with a dimension of 2. Let u and v be two non-adjacent vertices of the graph G. If u possesses a neighbor that is not contiguous to v, then this neighbor, together with u and v, forms a route P_4. This indicates that the diameter of G exceeds 2, which is infeasible. Any two non-adjacent vertices must possess a shared neighbor. Therefore, G is a complete bipartite graph. In the framework of a full bipartite graph, if G is either $K_{p,p}$ or $K_{n-1,1}$, no proof is necessary. Conversely, if G is $K_{m,n-m}$ with $m \neq 1$ and $m \neq \frac{n}{2}$, Theorem 5.1 posits that this graph possesses four distinct Laplacian harmonic eigenvalues. This concludes the proof in this instance.
(iv).Let G be a k-regular graph. Then in this case, we have $d_i = k$ for every i. The Laplacian Harmonic-matrix is expressed as $\mathcal{L}(G) = \frac{1}{k} L(G)$. This indicates that $\mathcal{L}(G)$ has three different eigenvalues if and only if $L(G)$ has three distinct eigenvalues. Based on the premise that regular graphs with three distinct Laplacian eigenvalues are precisely strongly regular graphs, the conclusion is drawn in this case. This completes the proof. □

Part (iii) of Theorem 5.8 indicates

the potential existence of additional non-bipartite graphs with a diameter of 2 that possess three separate Laplacian HI-eigenvalues. Consequently, we provide the following open problem.

Problem 1. *Determine all non-bipartite graphs with diameter 2 that have exactly three distinct Laplacian HI-eigenvalues.*

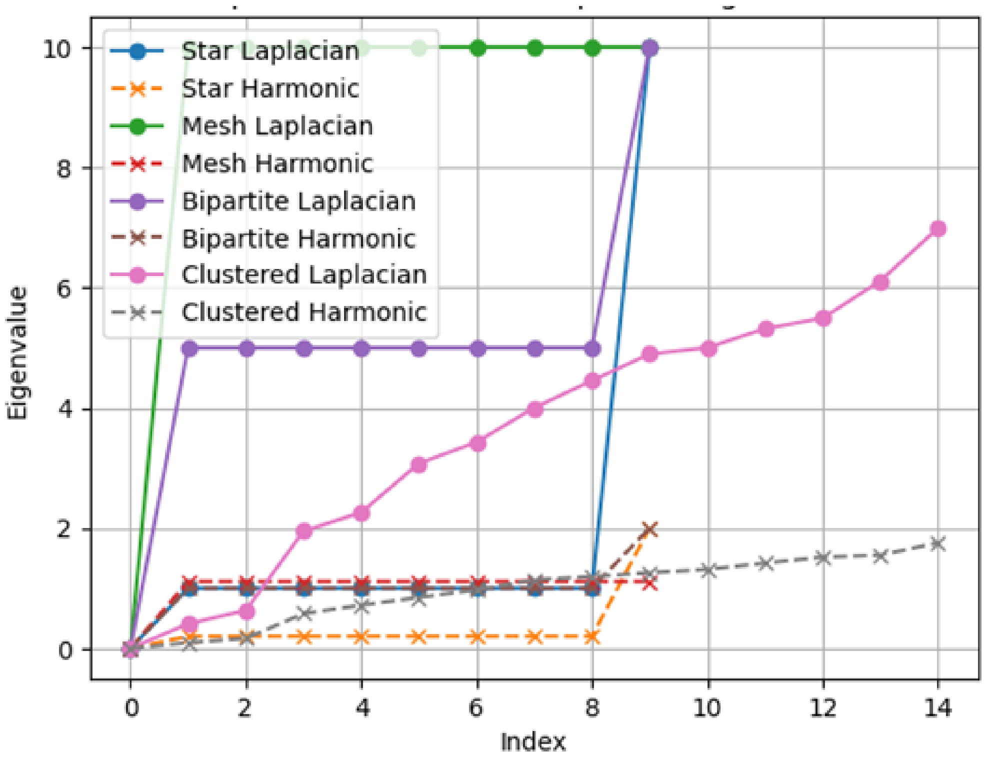
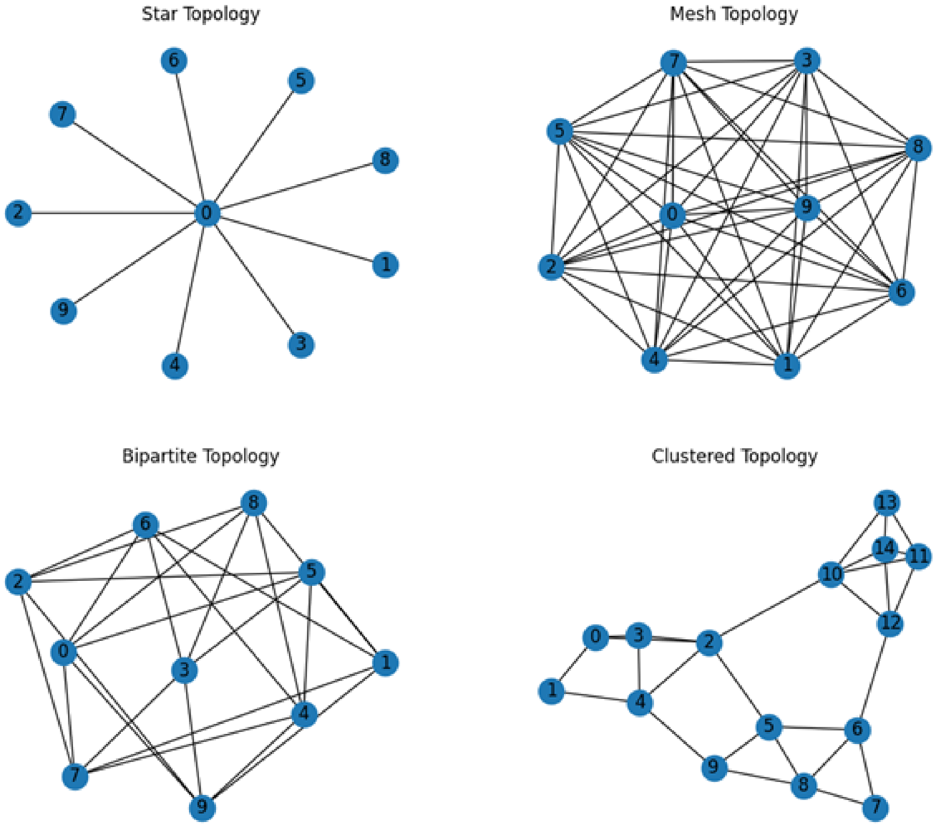


Figure 1: Laplacian vs Harmonic Laplacian Eigenvalues

The eigenvalue comparison graph illustrates clear spectral differences between the standard Laplacian and the Laplacian Harmonic matrices across all IoT topologies. In each case, the presence of at least one zero eigenvalue confirms that the generated networks are connected, ensuring communication among devices. However, the standard Laplacian exhibits a wider spread of eigenvalues with higher peaks, indicating that nodes with higher degrees (such as gateways in star topology) dominate the network structure. In contrast, the Laplacian Harmonic matrix produces a more compact and smoother eigenvalue distribution, reflecting a balanced contribution of all nodes. This normalization effect is particularly important in IoT environments, where heterogeneous devices coexist, as it reduces bias toward highly connected nodes and provides a more realistic representation of communication load distribution. Additionally, the second smallest eigenvalue (algebraic connectivity) varies across topologies, showing that mesh networks are the most robust, while star networks are more vulnerable to failures.



The topology visualization graphs further support these spectral observations by revealing structural differences among star, mesh, bipartite, and clustered networks. The star topology highlights a centralized structure prone to single-point failure, which is reflected in its lower spectral robustness, whereas the mesh topology demonstrates dense interconnections and high resilience, confirmed by its stronger eigenvalue profile. Bipartite networks show a balanced structure between two distinct node groups, commonly representing sensor-to-server communication in IoT systems, while clustered networks exhibit modular organization with strong intra-cluster and weak inter-cluster connections. The Laplacian Harmonic matrix effectively captures these structural characteristics by emphasizing balanced connectivity and local interactions, making it more suitable for analyzing real-world IoT deployments. Overall, the combined spectral and structural analysis demonstrates that the harmonic approach provides deeper insights into network stability, scalability, and energy-efficient communication in IoT systems.

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