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COMPLEX INTERVAL VALUED PYTHAGOREAN FUZZY ACZEL ALSINA AGGREGATION OPERATORS WITH APPLICATIONS

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Abstract
The common mathematical tool for combining several inputs into a single, unique result is an aggregation operator. The study introduces various aggregation operator (AOs) designed for complex interval valued Pythagorean fuzzy information. The complex interval valued Pythagorean fuzzy sets (CIVPyFS) which was created lately, proves to be a useful tool for expressing obscurity and ambiguities. The complex interval valued Pythagorean fuzzy sets have a wide range of applications in routine decision-making procedures because of their improved ability to handle uncertain circumstances compared to other fuzzy set theories. In this article, novel AOs are developed considering the advantages of the CIVPyFS to handle the multi-criteria decision-making challenges. The new AOs consider the relations between two input arguments. To improve the adaptability of the new AOs, this article incorporates the Aczel-Alsina (AA) operations. This study proposes the CIVPyF Aczel-Alsina Heronian mean (CIVPyFAAHM) operator, CIVPyF Aczel-Alsina geometric Heronian mean (CIVPyFAAGHM) operator which combines the Aczel-Alsina operational rules with the and Heronian mean/geometric Heronian mean operators. Various properties of the AOs are investigated. Further, weighted form these AOs are introduced. Then, we set up the MCDM technique using the two AOs that are suggested to solve MCDM problems under CIVPyFS environment. We next illustrate the efficacy and suitability of the predicted approach with a numerical example and compare it with other relevant MCDM strategies presently in existence in the CIVPyF information.

1. Introduction

Zadeh's initial proposal of fuzzy sets [1] established the groundwork for concept of intuitionistic fuzzy sets (IFSs). Atanassov [2] further advanced the conceptual framework of fuzzy sets, extending and progressing the foundational ideas in the field. In defining IFSs, one employs membership and non-membership functions, ensuring that their combined sum does not exceed one [3]. Yager [4] introduced the Pythagorean fuzzy set (PyFS), simplifying IFS by relaxing the constraint that limits quantity of the squares of membership and non-membership degrees to be below otherwise equivalent towards one. In 2013, Yager and Abbasov [5] demonstrated a significant advancement in the realm of PyFVs by introducing a representation utilizing complex numbers. Peng and Yang [6] made substantial contributions to the field of PyFSs and interval-valued PyFS. Zhang [7] enhanced PyFS into IVPyFS to address limitations and effectively manage ambiguous information. Ejegwa [8] introduced similarity measures and devised several AOs utilizing PyFSs to manage impressions and vague information effectively. Peng and Yang [9] contributed a suite of fresh AOs utilizing the IVPyF values (IVPyFVs), which they implemented in the context of group decision-making. Mu et al. [10] broadened the concept of IVPyFS through the utilization of Maclaurin symmetric mean operations. Yang et al. [11] expanded the scope of PyFSs by incorporating the functionalities of Frank power aggregation operators. Rahman and Ali [12] innovated new AOs along with their specialized instances, employing IVPyF values (IVPyFVs) in their development. Mishra et al. [13] applied the concept of IVPyF similarity measures to assess the optimal technology for energy production from waste materials. Ramot et al. [14] further expanded the scope by introducing Complex FSs, defining them as mathematical functions that express membership values in terms of complex numbers. Alkouri and Salleh [15] extended the theory of Complex FS to Complex IFS, incorporating the degree of (ῃ-MS) and providing a comprehensive description of their fundamental operations. Garg and Dimple [16] introduced CIVIFS to tackle time-periodic challenges. Ullah et al. [17] further advanced the field by developing the theory of CPyFS and delving into an exploration of their properties.

Triangular norms play a crucial role in aggregating fuzzy information, serving as robust tools to capture and manage uncertainty within fuzzy environments. In 1942, Menger [18] introduced the notions of T-NM and T-CNM, utilizing these concepts to address statistical measurements. Expanding upon the foundation laid by T-NM and T-CNM, Egbert [19] further developed the concept by defining Cartesian products and combining real numbers through addition. In their respective studies, Klement and Navara [20] delved into the Lukasiewicz T-NM logic within the framework of a fuzzy system, while Zeshui Xu [21] explored innovative aggregation operators (AOs) by applying algebraic addition and multiplication techniques IFSs. Garg [22] broadened the foundational assumptions of PyFSs, devising robust methodologies specifically tailored for the incorporation of Einstein models in fuzzy environments. Expanding on the IFS concept, Xu and Yager [23] presented the IF weighted geometric operator, strategically designed to tackle and solve a variety of everyday problems. Peng [24] introduced innovative methods for IVPyFS and proposed algorithms to address complexities in aggregating fuzzy information. Seikh and Mandal [25] elucidated appealing methodologies concerning Interval IFSs within the context of Dombi operations. Hussain et al. [26] developed several robust aggregation models and concurrently explored the application of vendor management systems. Khan et al. [27], [28] proposed methodologies for bipolar valued hesitant fuzzy systems and presented an appealing algorithm designed to address information loss during the aggregation process. Jana et al. [29] introduced aggregation operators for PyF Dombi weighted averaging and explored their properties by leveraging Dombi operations. Meanwhile, Zhang [30] provided reliable aggregation tools aimed at mitigating the influence of dubious information using Frank models. Xia et al. [31] expanded the scope of IFSs and introduced aggregation operators based on Archimedean TNM and TCNM. In contrast, Huang [32] introduced aggregation operators for IFSs and demonstrated their MADM technique for selecting suitable objectives, utilizing Hammer TNM and TCNM. Senapati and Chen [33] delved into the theory IVPyFSs, developing AOs for IVPyFSs using Hamacher TNM and TCNM. Ullah et al. [34] delved into the exploration of complex and uncertain information by employing the concepts of compelling models of Dombi operations within the framework of interval-valued T-spherical fuzzy sets.

The practice of multi-criteria decision-making (MCDM) represents a sophisticated methodology for effectively navigating complex and challenging data within real-world contexts. MCDM, integral to the decision-making sciences [35], [36], [37], [38], [39], [40], is a methodology capable of generating ranking grades for finite alternatives by evaluating the unique characteristics of each option. Senapati et al. [41] developed principles of AA tools. They further applied these tools in the context of IVIFSs, investigating their utility in solving a MADM problem. Hussain et al. [42]

developed various aggregation operators (AOs) for PyFSs, exploiting the operational laws of AA implements plus scrutinizing their fundamental properties. Additionally, Senapati [43] made significant contributions to the field by unveiling a new series of aggregation operators. This accomplishment was realized through the application of Aczel Alsina tools within the domain of illustrating fuzzy sets (FSs), highlighting the versatility and applicability of these tools across various fuzzy environments.

From the literature provided, it is evident that there is a lack of aggregation operators that have been integrated with AA operational laws, HM, and GHM operators. These aggregation operators possess the ability to mitigate the influence of outlier data while simultaneously accounting for the interdependency among input arguments. The objective of this study is to introduce aggregation operators with the specified attributes and develop an astute and strategic recommendation framework, facilitating the identification of the most suitable alternative strategy from a range of options. This will be achieved by formulating a plan that enables us to discern the most rational option, accommodating preferences for alternative methods in the selection process. The effectiveness of the proposed framework will be assessed by employing the CIVPyFAAWHM operator and the CIVPyFAAWGHM operator, both of which integrate AA operational rules into a CIVPyF framework. Incorporating AA operational laws into a CIVPyF context would essentially validate the efficacy of the proposed framework. Our central goal in this study is to reveal innovative aggregation operators grounded in the operational laws advanced by Aczel-Alsina. The comprehensive aims and objectives encompass:

- Examine various aggregation operators such as the Heronian Mean Operator (HMO), Weight Heronian Mean Operator (WHMO), Geometric Heronian Mean Operator (GHMO), and Weight Geometric Heronian Mean Operator (WGHMO).
- Define the (MCDM) model utilizing the mentioned (AOs).
- Apply the MCDM model to address and resolve problems involving multiple attributes.

After delving into various articles and research papers on fuzzy sets, we proceeded to assess and analyze their content. Moving forward, our subsequent phase involves a detailed examination of the operational laws proposed by Aczel-Alsina, with a focus on understanding their inherent properties. Our primary objective is to elevate simple Aczel-Alsina operations into a more complex domain, paving the way for the development of novel aggregation operators stemming from this transformation. We evaluate the MCDM model's performance by scrutinizing numerical examples and proceed to make comparisons with established models.

2. Preliminaries

Within this section, we discuss the fundamental definitions and result related to CIVPyFSs, exploring discussions on Aczel-Alsina TN and T-CN.

Definition 1 [44]. Let U as the universal set. A CIVPyFS symbolized as:

$$B_i = \{x, [P_{Bi}^L(x), P_{Bi}^U(x)], [Q_{Bi}^L(x), Q_{Bi}^U(x)]/x \epsilon U\}$$

Where $[P_{Bi}^L, P_{Bi}^U] : U \longrightarrow \{z_1^L, z_1^U : z_1^L, z_1^U \epsilon \mathbb{C} : |z_1^L|, |z_1^U| \leq 1\}$ and $[Q_{Bi}^L, Q_{Bi}^U] : U \longrightarrow \{z_2^L, z_2^U : z_2^L, z_2^U \epsilon \mathbb{C} : |z_2^L|, |z_2^U| \leq 1\}$.

Such that $P_{Bi}^L = z_1^L = \gamma_i^L(x) e^{2\pi\iota\theta_{\gamma i}^L(x)}$, $P_{Bi}^U = z_1^U = \gamma_i^U(x) e^{2\pi\iota\theta_{\gamma i}^U(x)}$ and $Q_{Bi}^L = z_1^L = \omega_i^L(x) e^{2\pi\iota{\theta'}_{\omega i}^L(x)}$, $Q_{Bi}^U = z_1^U = \omega_i^U(x) e^{2\pi\iota{\theta'}_{\omega i}^U(x)}$.

Satisfying the condition $0 \leq \left(\gamma_{Bi}^U(x)\right)^2 + \left(\omega_{Bi}^U(x)\right)^2 \leq 1$ and $0 \leq \left(\theta_{Bi}^U(x)\right)^2 + \left({\theta'}_{Bi}^U(x)\right)^2 \leq 1$.

In mathematical terms, the CIVPyFS B_i defined on the universal set U can be expressed or symbolized as: $B_i = \left(x, [\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota\left[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)\right]}/x \in \mathrm{U}\right)$. The amplitude terms are defined by $([\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)], [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)]) \subset [0,1]$. Moreover, the real-valued phase terms are confined to the specified interval are given by $\left[\theta_{Bi}^{L}(x), \theta_{Bi}^{U}(x)\right], \left[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)\right] \subset [0,1]$ and the condition are as follows:

$$\left(\gamma_{Bi}^{U}(x)\right)^2 + \left(\omega_{Bi}^{U}(x)\right)^2 \le 1, \left(\theta_{Bi}^{U}(x)\right)^2 + \left(\theta'^{U}_{Bi}(x)\right)^2 \le 1.$$

Moreover,

$B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota\left[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)\right]}\rangle$ is called CIVPyFN.

Definition 2. For any CIVPyFN, $\mathfrak{K} = \langle[\gamma_{\mathfrak{K}}^{L}, \gamma_{\mathfrak{K}}^{U}].e^{2\pi\iota[\theta_{\mathfrak{K}}^{L},\theta_{\mathfrak{K}}^{U}]}, [\omega_{\mathfrak{K}}^{L}, \omega_{\mathfrak{K}}^{U}].e^{2\pi\iota\left[\theta'^{L}_{\mathfrak{K}},\theta'^{U}_{\mathfrak{K}}\right]}\rangle$, we define the score function $(\mathcal{S})$ as:

$$\ddot{\mathcal{S}}(\mathfrak{K}) = \frac{1}{2}\langle\left(\gamma^{L}(x)\right)^2 + \left(\gamma^{U}(x)\right)^2 + \left(\theta^{L}(x)\right)^2 + \left(\theta^{U}(x)\right)^2 - \left(\omega^{L}(x)\right)^2 - \left(\omega^{U}(x)\right)^2 - \left(\theta'^{L}(x)\right)^2 - \left(\theta'^{U}(x)\right)^2\rangle$$

It is clear that $\ddot{\mathcal{S}}(B) \epsilon [-1,1]$.

And for any CIVPyFN, $\mathfrak{K} = \langle[\gamma_{\mathfrak{K}}^{L}, \gamma_{\mathfrak{K}}^{U}].e^{2\pi\iota[\theta_{\mathfrak{K}}^{L},\theta_{\mathfrak{K}}^{U}]}, [\omega_{\mathfrak{K}}^{L}, \omega_{\mathfrak{K}}^{U}].e^{2\pi\iota\left[\theta'^{L}_{\mathfrak{K}},\theta'^{U}_{\mathfrak{K}}\right]}\rangle$, we define the accuracy function $(\mathcal{A})$ as:

$$\ddot{\mathcal{A}}(\mathfrak{K}) = \frac{1}{2}\langle\left(\gamma^{\mathrm{L}}(\mathrm{x})\right)^2 + \left(\gamma^{\mathrm{U}}(\mathrm{x})\right)^2 + \left(\theta^{\mathrm{L}}(\mathrm{x})\right)^2 + \left(\theta^{\mathrm{U}}(\mathrm{x})\right)^2 + \left(\omega^{\mathrm{L}}(\mathrm{x})\right)^2 + \left(\omega^{\mathrm{U}}(\mathrm{x})\right)^2 + \left(\theta'^{\mathrm{L}}(\mathrm{x})\right)^2 + \left(\theta'^{\mathrm{U}}(\mathrm{x})\right)^2\rangle \epsilon [0,1]$$

If $\mathfrak{K}_1 = \langle[\gamma_{\mathfrak{K}1}^{L}, \gamma_{\mathfrak{K}1}^{U}].e^{2\pi\iota[\theta_{\mathfrak{K}1}^{L},\theta_{\mathfrak{K}1}^{U}]}, [\omega_{\mathfrak{K}1}^{L}, \omega_{\mathfrak{K}1}^{U}].e^{2\pi\iota\left[\theta'^{L}_{\mathfrak{K}1},\theta'^{U}_{\mathfrak{K}1}\right]}\rangle$ *and*,

$\mathfrak{K}_2 = \langle[\gamma_{\mathfrak{K}2}^{L}, \gamma_{\mathfrak{K}2}^{U}].e^{2\pi\iota[\theta_{\mathfrak{K}2}^{L},\theta_{\mathfrak{K}2}^{U}]}, [\omega_{\mathfrak{K}2}^{L}, \omega_{\mathfrak{K}2}^{U}].e^{2\pi\iota\left[\theta'^{L}_{\mathfrak{K}2},\theta'^{U}_{\mathfrak{K}2}\right]}\rangle$ are two CIVPyFVs. Then,

- $\mathcal{S}(\mathfrak{K}_1) < \mathcal{S}(\mathfrak{K}_2)$ if $\mathfrak{K}_1 < \mathfrak{K}_2$
- $\mathcal{S}(\mathfrak{K}_1) > \mathcal{S}(\mathfrak{K}_2)$ if $\mathfrak{K}_1 < \mathfrak{K}_2$
- $\mathcal{S}(\mathfrak{K}_1) = \mathcal{S}(\mathfrak{K}_2)$, then:
- $\mathcal{A}(\mathfrak{K}_1) < \mathcal{A}(\mathfrak{K}_2)$ if $\mathfrak{K}_1 < \mathfrak{K}_2$
- $\mathcal{A}(\mathfrak{K}_1) > \mathcal{A}(\mathfrak{K}_2)$ if $\mathfrak{K}_1 > \mathfrak{K}_2$
- $\mathcal{A}(\mathfrak{K}_1) = \mathcal{A}(\mathfrak{K}_2)$ if $\mathfrak{K}_{1_1} = \mathfrak{K}_2$

Definition 3 [49]. The concept of Aczel Alsina TN is in the following form:

$$\mathfrak{G}_{\sigma}^{\mathscr{s}}(\mathfrak{a}, \mathfrak{b}) = \begin{Bmatrix} \mathfrak{E}_{\mathfrak{D}}(\mathfrak{a}, \mathfrak{b})\ if\ \mathscr{s} = 0 & \\ \min(\mathfrak{a}, \mathfrak{b})\ if\ \mathscr{s} = \infty & \forall, 0 \le \mathscr{s} \le +\infty \\ e^{-((-log\mathfrak{a})^{\mathscr{s}} + (-log\mathfrak{b})^{\mathscr{s}})^{\frac{1}{\mathscr{s}}}} & otherwise \end{Bmatrix}.$$

Definition 4 [49]. The concept of Aczel Alsina T-CN is in the following form:

$$\mathfrak{N}_{\sigma}^{s}(\mathfrak{a},\mathfrak{b}) = \begin{Bmatrix} \mathfrak{N}_{\mathfrak{D}}(\mathfrak{a},\mathfrak{b})\ if\ s=0 & \\ \max(\mathfrak{a},\mathfrak{b})\ if\ s=\infty & \forall,0 \le s \le +\infty \\ 1-e^{-((-log\mathfrak{a})^{s}+(-log\mathfrak{b})^{s})^{\frac{1}{s}}} & otherwise \end{Bmatrix}.$$

Definition 5 [50]. Let $B_i = \langle[\gamma_{Bi}^L(x), \gamma_{Bi}^U(x)].e^{2\pi\iota[\theta_{Bi}^L(x),\theta_{Bi}^U(x)]}, [\omega_{Bi}^L(x), \omega_{Bi}^U(x)].e^{2\pi\iota[\theta'^L_{Bi}(x),\theta'^U_{Bi}(x)]}\rangle$ be the family of real no's with $\mathfrak{p} > 0, \mathfrak{q} > 0$, then, the HM operator is defined as:

$$HM(B_1, B_2, B_3, \ldots\ldots, B_\eta) = \left(\frac{2}{\eta(\eta+1)}\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(B_i^{\mathfrak{p}}.B_j^{\mathfrak{q}}\right)\right)^{\frac{1}{\mathfrak{p}+\mathfrak{q}}}.$$

HM requirement satisfy the following conditions:

1) If all $\sigma_i = 0, \forall i$,then, the $HM^{\mathfrak{p},\mathfrak{q}}(0,0,0 \ldots,0) = 0$.
2) If all $\sigma_i = \sigma, \forall i$,then, the $HM^{\mathfrak{p},\mathfrak{q}}(\sigma_1, \sigma_2, \ldots, \sigma_\eta) = \sigma$.
3) If all $\sigma_i \ge \daleth_i, \forall i$,then, the $HM^{\mathfrak{p},\mathfrak{q}}(\sigma_1, \sigma_2, \ldots, \sigma_\eta) \ge HM^{\mathfrak{p},\mathfrak{q}}(\daleth_1, \daleth_2, \ldots, \daleth_\eta)$.
4) $B_i^- \le HM^{\mathfrak{p},\mathfrak{q}}(B_1, B_2, B_3, \ldots\ldots, B_\eta) \le B_i^+$.

Definition 6 [50]. Let $B_i = \langle[\gamma_{Bi}^L(x), \gamma_{Bi}^U(x)].e^{2\pi\iota[\theta_{Bi}^L(x),\theta_{Bi}^U(x)]}, [\omega_{Bi}^L(x), \omega_{Bi}^U(x)].e^{2\pi\iota[\theta'^L_{Bi}(x),\theta'^U_{Bi}(x)]}\rangle$ be the family of real no's with $\mathfrak{p} > 0, \mathfrak{q} > 0$, then, the GHM operator is defined as:

$$GHM(B_1, B_2, B_3, \ldots\ldots, B_\eta) = \left(\frac{1}{\mathfrak{p}+\mathfrak{q}}\otimes_{\substack{i=1 \\ j=1}}^{\eta}\mathfrak{p}B_i \oplus \mathfrak{q}B_j\right)^{\frac{2}{\eta(\eta+1)}}.$$

GHM requirement satisfy the following conditions:

1. If all $\sigma_i = 0, \forall i$,then, the $GHM^{\mathfrak{p},\mathfrak{q}}(0,0,0 \ldots,0) = 0$.
2. If all $\sigma_i = \sigma, \forall i$,then, the $GHM^{\mathfrak{p},\mathfrak{q}}(\sigma_1, \sigma_2, \ldots, \sigma_\eta) = \sigma$.
3. If all $\sigma_i \ge \daleth_i, \forall i$,then, the $GHM^{\mathfrak{p},\mathfrak{q}}(\sigma_1, \sigma_2, \ldots, \sigma_\eta) \ge GHM^{\mathfrak{p},\mathfrak{q}}(\daleth_1, \daleth_2, \ldots, \daleth_\eta)$.
4. $B_i^- \le GHM^{\mathfrak{p},\mathfrak{q}}(B_1, B_2, B_3, \ldots\ldots, B_\eta) \le B_i^+$.

2.1. Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Operators

Within the CIVPyF information system, we investigate the role of Aczel Alsina operations, encompassing algebraic sum, product, scalar multiplication, and power.

Definition 7. Let $B = \langle[\gamma_B^L(x), \gamma_B^U(x)].e^{2\pi\iota[\theta_B^L(x),\theta_B^U(x)]}, [\omega_B^L(x), \omega_B^U(x)].e^{2\pi\iota[\theta'^L_B(x),\theta'^U_B(x)]}\rangle$,

$B_1 = \langle[\gamma_{B1}^L(x), \gamma_{B1}^U(x)].e^{2\pi\iota[\theta_{B1}^L(x),\theta_{B1}^U(x)]}, [\omega_{B1}^L(x), \omega_{B1}^U(x)].e^{2\pi\iota[\theta'^L_{B1}(x),\theta'^U_{B1}(x)]}\rangle$ *and*

$B_2 = [\gamma_{B2}^L(x), \gamma_{B2}^U(x)].e^{2\pi\iota[\theta_{B2}^L(x),\theta_{B2}^U(x)]}, [\omega_{B2}^L(x), \omega_{B2}^U(x)].e^{2\pi\iota[\theta'^L_{B2}(x),\theta'^U_{B2}(x)]}$ be the three CIVPyFNs , and $\eta > 0, \text{Å} \ge 1$ be any real numbers. Then, some basic operations of CIVPyFNs are given as:

$$\eta B=\left\langle\left[\sqrt{1-e^{-\left(\eta\left(-ln\left(1-(\gamma_B^L)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\eta\left(-ln\left(1-(\gamma_B^U)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\eta\left(-ln\left(1-(\theta_B^L)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\eta\left(-ln\left(1-(\theta_B^U)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left.\left[\sqrt{e^{-\left(\eta\left(-ln(\omega_B^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\eta\left(-ln(\omega_B^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\eta\left(-ln(\theta'^L_B)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\eta\left(-ln(\theta'^U_B)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

$$B^{\eta}=\left\langle\left[\sqrt{e^{-\left(\eta\left(-\ln(\gamma_B^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\eta\left(-\ln(\gamma_B^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\eta\left(-\ln(\theta_B^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\eta\left(-\ln(\theta_B^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left.\left[\sqrt{1-e^{-\left(\eta\left(-\ln\left(1-(\omega_B^L)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\eta\left(-\ln\left(1-(\omega_B^U)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\eta\left(-\ln\left(1-(\theta'^L_B)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\eta\left(-\ln\left(1-(\theta'^U_B)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

$$B_1\oplus B_2=\left\langle\begin{array}{l}\left[\sqrt{1-e^{-\left(\left(-\ln\left(1-(\gamma_{B1}^L)^2\right)\right)^{\mathring{A}}+\left(-\ln\left(1-(\gamma_{B2}^L)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(-\ln\left(1-(\gamma_{B1}^U)^2\right)\right)^{\mathring{A}}+\left(-\ln\left(1-(\gamma_{B2}^U)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]\\ .e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(-\ln\left(1-(\theta_{B1}^L)^2\right)\right)^{\mathring{A}}+\left(-\ln\left(1-(\theta_{B2}^L)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(-\ln\left(1-(\theta_{B1}^U)^2\right)\right)^{\mathring{A}}+\left(-\ln\left(1-(\theta_{B2}^U)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\end{array}\right.$$

$$\left.\left[\sqrt{e^{-\left(\left(-\ln(\omega_{B1}^L)^2\right)^{\mathring{A}}+\left(-\ln(\omega_{B2}^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(-\ln(\omega_{B1}^U)^2\right)^{\mathring{A}}+\left(-\ln(\omega_{B2}^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\left(-\ln(\theta'^L_{B1})^2\right)^{\mathring{A}}+\left(-\ln(\theta'^L_{B2})^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(-\ln(\theta'^U_{B1})^2\right)^{\mathring{A}}+\left(-\ln(\theta'^U_{B2})^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

$$B_1\otimes B_2=\left\langle\left[\sqrt{e^{-\left(\left(-\ln(\gamma_{B1}^L)^2\right)^{\mathring{A}}+\left(-\ln(\gamma_{B2}^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(-\ln(\gamma_{B1}^U)^2\right)^{\mathring{A}}+\left(-\ln(\gamma_{B2}^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\left(-\ln(\theta_{B1}^L)^2\right)^{\mathring{A}}+\left(-\ln(\theta_{B2}^L)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(-\ln(\theta_{B1}^U)^2\right)^{\mathring{A}}+\left(-\ln(\theta_{B2}^U)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left[\sqrt{1-e^{-\left(\left(-\ln\left(1-\left(\omega_{B1}^{L}\right)^{2}\right)\right)^{\mathring{A}}+\left(-\ln\left(1-\left(\omega_{B2}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(-\ln\left(1-\left(\omega_{B1}^{U}\right)^{2}\right)\right)^{\mathring{A}}+\left(-\ln\left(1-\left(\omega_{B2}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(-\ln\left(1-\left(\theta'^{L}_{B1}\right)^{2}\right)\right)^{\mathring{A}}+\left(-\ln\left(1-\left(\theta'^{L}_{B2}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(-\ln\left(1-\left(\theta'^{U}_{B1}\right)^{2}\right)\right)^{\mathring{A}}+\left(-\ln\left(1-\left(\theta'^{U}_{B2}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

3. Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Heronian Mean Operators

In this section, based on the AA operational laws, several AOs, such as complex interval-valued Pythagorean fuzzy Aczel-Alsina Heronian mean (CIVPyFAAHM) operators, complex interval-valued Pythagorean fuzzy Aczel-Alsina weighted Heronian mean (CIVPyFAAHM) operators are presented along with their fundamental properties.

3.1. The Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Heronian Mean Operators

In this subsection, the CIVPyFAAHM operators are introduced along with their fundamental properties.

Definition 8. Let $B_i = \langle[\gamma_{Bi}^{L}(x),\gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]},[\omega_{Bi}^{L}(x),\omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$ be the family of CIVPyFNs and $\text{ꝑ}>0,\text{ꝗ}>0$. Then, CIVPyFAAHM is defined as follows:

$$CIVPyFAAHM^{\text{ꝑ},\text{ꝗ}}(B_1,B_2,B_3,\ldots\ldots,B_{\text{ŋ}})=\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(B_i^{\text{ꝑ}}\times B_j^{\text{ꝗ}}\right)\right)^{\frac{1}{\text{ꝑ}+\text{ꝗ}}}$$

Theorem 1. Let $B_i = \langle[\gamma_{Bi}^{L}(x),\gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]},[\omega_{Bi}^{L}(x),\omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$ be the family of CIVPyFNs and $\text{ꝑ}>0,\text{ꝗ}>0$. Then, the aggregated value of CIVPyFAAHM is defined as follows:

$$CIVPyFAAHM^{\text{ꝑ},\text{ꝗ}}(B_1,B_2,B_3,\ldots\ldots,B_{\text{ŋ}})=\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(B_i^{\text{ꝑ}}\times B_j^{\text{ꝗ}}\right)\right)^{\frac{1}{\text{ꝑ}+\text{ꝗ}}}$$

$$=\Bigg\langle\left[\sqrt{e^{-\left(\left(\frac{1}{\text{ꝑ}+\text{ꝗ}}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(\text{ꝑ}\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(\text{ꝗ}\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(\frac{1}{\text{ꝑ}+\text{ꝗ}}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(\text{ꝑ}\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)+\left(\text{ꝗ}\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{\text{ꝑ}+\text{ꝗ}}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(\text{ꝑ}\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(\text{ꝗ}\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(\frac{1}{\text{ꝑ}+\text{ꝗ}}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\sum_{i=1}^{\text{ŋ}}\sum_{j=i}^{\text{ŋ}}\left(\text{ꝑ}\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)+\left(\text{ꝗ}\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left.\left[\sqrt{1-e^{-\left(\left(\frac{1}{\mathfrak{p}+\mathfrak{q}}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(\mathfrak{p}\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+\mathfrak{q}\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{\mathfrak{p}+\mathfrak{q}}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(\mathfrak{p}\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+\mathfrak{q}\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{\mathfrak{p}+\mathfrak{q}}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(\mathfrak{p}\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+\mathfrak{q}\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{\mathfrak{p}+\mathfrak{q}}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(\mathfrak{p}\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+\mathfrak{q}\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

Proof: Let $B_i = \left\langle\left[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)\right]}, \left[\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)\right]}\right\rangle$, i=1,2,3,………,$\eta$ be the family of CIVPyFNs and $\mathfrak{p} > 0, \mathfrak{q} > 0$, we have

$$B_i^{\mathfrak{p}} = \left\langle\left[\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left.\left[\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

$$B_j^{\mathfrak{q}} = \left\langle\left[\sqrt{e^{-\left(\mathfrak{q}\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{q}\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\mathfrak{q}\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{q}\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left.\left[\sqrt{1-e^{-\left(\mathfrak{q}\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{q}\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\mathfrak{q}\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{q}\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

Now,

$$B_i^{p} \otimes B_j^{q} = \Bigg\langle \left[\sqrt{e^{-\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right] . e^{2\pi\iota\left[\sqrt{e^{-\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right] . e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Let $z = 1 - e^{-\left(p\left(-\ln\left(1-(\omega_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj})^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}$, $1 - z = e^{-\left(p\left(-\ln\left(1-(\omega_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj})^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}$

$\ln(1-z) = \ln e^{-\left(p\left(-\ln\left(1-(\omega_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj})^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}$ and $\ln(1-z) = -\left(p\left(-\ln\left(1-(\omega_{Bi})^2\right)\right)^{\mathring{A}} + q\left(-\ln\left(1-(\omega_{Bj})^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}$.

Hence,

$$\sum_{i=1}^{\eta}\sum_{j=i}^{\eta} B_i^{p} \otimes B_j^{q} = \Bigg\langle \left[\sqrt{1-e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right] . e^{2\pi\iota\left[\sqrt{1-e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right] . e^{2\pi\iota\left[\sqrt{e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Therefore,

$$\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(B_i^{p} \otimes B_j^{q}\right)\right)\right) =$$

$$\left\langle \begin{array}{l} \left[\sqrt{1-e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}+q\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^2\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}, \\ \left[\sqrt{e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]} \end{array} \right\rangle.$$

Now let

$$z = \sqrt{1-e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-\gamma_{Bi}{}^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}\right)\right)^{\mathring{A}}}}$$

$z^2 = 1 - e^{-x}$

$\ln(1 - z^2) = \ln e^{-x}$

Also let $z = 1 - \left(e^{-\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-\omega_{Bi}{}^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}\right)\right)^{\mathring{A}}}\right)^2$

$z = 1 - (e^{-x})^2$

$1 - z = e^{-2x}$

$\ln(1 - z) = \ln e^{-2x}$

$\ln(1 - z) = -2x$

$\frac{1}{2}\left(\ln(1 - z)\right) = -x$

$\ln\sqrt{(1 - z)} = -x$

So we have,

$$\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(B_i^{p}\otimes B_j^{q}\right)\right)\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right]},\right.$$

$$\left.\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right]}\right\rangle.$$

$$CIVPyFAAHM^{p,q}\left(B_1,B_2,B_3,\ldots\ldots,B_\eta\right)=$$

$$\left\langle\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\text{Å}}+\left(q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\text{Å}}\right)\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right]},\right.$$

$$\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\omega_{Bi}^{L})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj}^{L})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\omega_{Bi}^{U})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj}^{U})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\theta'^{L}_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta'^{L}_{Bj})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\theta'^{U}_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta'^{U}_{Bj})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Big\rangle.$$

Theorem 2 (Idempotency). Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, $i = 1,2,3,,\ldots\ldots\ldots,\eta$ be the collection of identical CIVPyFNs. Then,

$$CIVPyFAAHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_\eta) = B.$$

Proof: Since $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle = B =$ $\langle[\gamma_{B}^{L}(x), \gamma_{B}^{U}(x)].e^{2\pi\iota[\theta_{B}^{L}(x),\theta_{B}^{U}(x)]}, [\omega_{B}^{L}(x), \omega_{B}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{B}(x),\theta'^{U}_{B}(x)]}\rangle$, $i = 1,2,3,,\ldots\ldots\ldots,\eta$. Then $CIVPyFAAHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_\eta) = CIVPyFAAHM^{p,q}(B, B, B, \ldots\ldots, B)$.

So,

$$=\Big\langle\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln(\gamma_{Bi}^{L})^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln(\gamma_{Bj}^{L})^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln(\gamma_{Bi}^{U})^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln(\gamma_{Bj}^{U})^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln(\theta_{Bi}^{L})^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln(\theta_{Bj}^{L})^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln(\theta_{Bi}^{U})^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln(\theta_{Bj}^{U})^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\omega_{Bi}^{L})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj}^{L})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\omega_{Bi}^{U})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\omega_{Bj}^{U})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\theta'^{L}_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta'^{L}_{Bj})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-(\theta'^{U}_{Bi})^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta'^{U}_{Bj})^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Big\rangle.$$

$$
= \Bigg\langle \left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]
$$

$$
. e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]},
$$

$$
\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]
$$

$$
. e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\text{ŋ}(\text{ŋ}+1)}\right)\left(\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]} \Bigg\rangle.
$$

$$
= \Bigg\langle \left[\sqrt{e^{-\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}} \right]
$$

$$
. e^{2\pi\iota\left[\sqrt{e^{-\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}} \right]},
$$

$$
\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}} \right]
$$

$$
. e^{2\pi\iota \sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}} \Bigg\rangle.
$$

$$
= \left\langle \left[\gamma_{B}^{L}(x), \gamma_{B}^{U}(x)\right]. e^{2\pi\iota\left[\theta_{B}^{L}(x), \theta_{B}^{U}(x)\right]}, \left[\omega_{B}^{L}(x), \omega_{B}^{U}(x)\right]. e^{2\pi\iota\left[\theta'^{L}_{B}(x), \theta'^{U}_{B}(x)\right]} \right\rangle.
$$

$$
= B.
$$

Theorem 3 (Monotonicity). Let $B_i = \left\langle \left[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)\right]. e^{2\pi\iota\left[\theta_{Bi}^{L}(x), \theta_{Bi}^{U}(x)\right]}, \left[\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)\right]. e^{2\pi\iota\left[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)\right]} \right\rangle$ and $C_i = \left\langle \left[\gamma_{Ci}^{L}(x), \gamma_{Ci}^{U}(x)\right]. e^{2\pi\iota\left[\theta_{Ci}^{L}(x), \theta_{Ci}^{U}(x)\right]}, \left[\omega_{Ci}^{L}(x), \omega_{Ci}^{U}(x)\right]. e^{2\pi\iota\left[\theta'^{L}_{Ci}(x), \theta'^{U}_{Ci}(x)\right]} \right\rangle$, $i = 1,2,3, \ldots \ldots, \text{ŋ}$ to be the two CIVPyFNs, if $B_i \leq C_i, \forall\, i$. Then,

$$CIVPyFAAHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_\eta) \leq CIVPyFAAHM^{p,q}(C_1, C_2, C_3, \ldots\ldots, C_\eta).$$

Proof: Let $B_i = \langle [\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x), \theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)]} \rangle$, $i = 1,2,3, \ldots\ldots\ldots, \eta$ be the family CIVPyFNs and $B_i \leq C_i, \forall\, i$. Then we show that
$CIVPyFAAHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_\eta) \leq CIVPyFAAHM^{p,q}(C_1, C_2, C_3, \ldots\ldots, C_\eta)$.

So, we have

$$= \left\langle \begin{array}{l} \left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}, \sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}} \right] \\ .e^{2\pi\iota\left[e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}, e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}} \right]}, \\ \left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right] \\ .e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]} \end{array} \right\rangle.$$

$$\leq \left\langle \begin{array}{l} \left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Ci}^{L}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Cj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}, \sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\gamma_{Ci}^{U}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\gamma_{Cj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}} \right] \\ .e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Ci}^{L}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Cj}^{L}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=i}^{\eta}\left(p\left(-\ln\left(\theta_{Ci}^{U}\right)^2\right)^{\mathring{A}}\right)+\left(q\left(-\ln\left(\theta_{Cj}^{U}\right)^2\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}} \right]}, \end{array} \right.$$

$$\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=i}^{ŋ}\left(p\left(-\ln\left(1-(\omega_{Ci}^{L})^{2}\right)\right)^{Å}+q\left(-\ln\left(1-(\omega_{Cj}^{L})^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=i}^{ŋ}\left(p\left(-\ln\left(1-(\omega_{Ci}^{U})^{2}\right)\right)^{Å}+q\left(-\ln\left(1-(\omega_{Cj}^{U})^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=i}^{ŋ}\left(p\left(-\ln\left(1-(\theta'^{L}_{Ci})^{2}\right)\right)^{Å}+q\left(-\ln\left(1-(\theta'^{L}_{Cj})^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=i}^{ŋ}\left(p\left(-\ln\left(1-(\theta'^{U}_{Ci})^{2}\right)\right)^{Å}+q\left(-\ln\left(1-(\theta'^{U}_{Cj})^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}\Bigg\rangle.$$

So, $\text{CIVPyFAAHM}^{p,q}(B_1,B_2,B_3,\ldots\ldots,B_ŋ)\leq\text{CIVPyFAAHM}^{p,q}(C_1,C_2,C_3,\ldots\ldots,C_ŋ)$.

Theorem 4 (Boundedness). Let $B_i=\left\langle\left[\gamma_{Bi}^{L}(x),\gamma_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)\right]},\left[\omega_{Bi}^{L}(x),\omega_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)\right]}\right\rangle$, $i=1,2,3,\ldots\ldots\ldots,ŋ$ be the family of CIVPyFNs, and $B_i^{-}=\min(B_1,B_2,B_3,\ldots\ldots,B_ŋ)$ and $B_i^{+}=\max(B_1,B_2,B_3,\ldots\ldots,B_ŋ)$, then,

$$B_i^{-}\leq\text{CIVPyFAAHM}^{p,q}(B_1,B_2,B_3,\ldots\ldots,B_ŋ)\leq B_i^{+}.$$

Proof: Suppose that $B_i=\left\langle\left[\gamma_{Bi}^{L}(x),\gamma_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)\right]},\left[\omega_{Bi}^{L}(x),\omega_{Bi}^{U}(x)\right].e^{2\pi\iota\left[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)\right]}\right\rangle$, $i=1,2,3,\ldots\ldots\ldots,ŋ$ be the family of CIVPyFVs, and $B_i^{-}=\min(B_1,B_2,B_3,\ldots\ldots,B_ŋ)$ of and $B_i^{+}=\max(B_1,B_2,B_3,\ldots\ldots,B_ŋ)$. So,

$$=\left\langle\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\gamma_{Bi}^{L}-)^{2}\right)^{Å}\right)+\left(q\left(-\ln(\gamma_{Bj}^{L}-)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\gamma_{Bi}^{U}-)^{2}\right)^{Å}\right)+\left(q\left(-\ln(\gamma_{Bj}^{U}-)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right.$$
$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\theta_{Bi}^{L}-)^{2}\right)^{Å}\right)+\left(q\left(-\ln(\theta_{Bj}^{L}-)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\theta_{Bi}^{U}-)^{2}\right)^{Å}\right)+\left(q\left(-\ln(\theta_{Bj}^{U}-)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}$$

$$\leq\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\gamma_{Bi}^{L})^{2}\right)^{Å}\right)+\left(q\left(-\ln(\gamma_{Bj}^{L})^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\gamma_{Bi}^{U})^{2}\right)^{Å}\right)+\left(q\left(-\ln(\gamma_{Bj}^{U})^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\theta_{Bi}^{L})^{2}\right)^{Å}\right)+\left(q\left(-\ln(\theta_{Bj}^{L})^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{ŋ(ŋ+1)}\right)\left(\sum_{i=1}^{ŋ}\sum_{j=1}^{ŋ}\left(p\left(-\ln(\theta_{Bi}^{U})^{2}\right)^{Å}\right)+\left(q\left(-\ln(\theta_{Bj}^{U})^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}$$

$$\leq\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi+}^{L}\right)^{2}\right)^{Å}\right)+\left(q\left(-\ln\left(\gamma_{Bj+}^{L}\right)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(\gamma_{Bi+}^{U}\right)^{2}\right)^{Å}\right)+\left(q\left(-\ln\left(\gamma_{Bj+}^{U}\right)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(\theta_{Bi+}^{L}\right)^{2}\right)^{Å}\right)+\left(q\left(-\ln\left(\theta_{Bj+}^{L}\right)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(\theta_{Bi+}^{U}\right)^{2}\right)^{Å}\right)+\left(q\left(-\ln\left(\theta_{Bj+}^{U}\right)^{2}\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}$$

$$\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi-}^{L}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj-}^{L}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi-}^{U}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj-}^{U}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi-}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj-}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi-}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj-}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}$$

$$\leq\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}$$

$$\leq\left[\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi+}^{L}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj+}^{L}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\left(\frac{1}{p+q}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\sum_{i=1}^{\eta}\sum_{j=1}^{\eta}\left(p\left(-\ln\left(1-\left(\omega_{Bi+}^{U}\right)^{2}\right)\right)^{Å}+q\left(-\ln\left(1-\left(\omega_{Bj+}^{U}\right)^{2}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$. e^{2\pi\iota\left[\sqrt{1-e^{-\left(\left(\frac{1}{\wp+\varrho}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\Sigma_{i=1}^{\eta}\Sigma_{j=1}^{\eta}\left(\wp\left(-\ln\left(1-\left(\theta'^{L}_{Bi+}\right)^{2}\right)\right)^{\mathring{A}}+\varrho\left(-\ln\left(1-\left(\theta'^{L}_{Bj+}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\left(\frac{1}{\wp+\varrho}\right)\left(\left(\frac{2}{\eta(\eta+1)}\right)\left(\Sigma_{i=1}^{\eta}\Sigma_{j=1}^{\eta}\left(\wp\left(-\ln\left(1-\left(\theta'^{U}_{Bi+}\right)^{2}\right)\right)^{\mathring{A}}+\varrho\left(-\ln\left(1-\left(\theta'^{U}_{Bj+}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\rangle.$$

Hence

$$B_i^{-} \leq \mathrm{CIVPyFAAHM}^{\wp,\varrho}(B_1, B_2, B_3, \ldots\ldots, B_\eta) \leq B_i^{+}.$$

Theorem 5 (Commutativity). Let $B_1{}', B_2{}', B_3{}', \ldots\ldots, B_\eta{}'$ be any permutation of $B_1, B_2, B_3, \ldots\ldots, B_\eta$. Then,

$$\mathrm{CIVPyFAAHM}^{\wp,\varrho}(B_1{}', B_2{}', B_3{}', \ldots\ldots, B_\eta{}') = \mathrm{CIVPyFAAHM}^{\wp,\varrho}(B_1, B_2, B_3, \ldots\ldots, B_\eta).$$

Proof: Let $B_1{}', B_2{}', B_3{}', \ldots\ldots, B_\eta{}'$ be any permutation of $B_1, B_2, B_3, \ldots\ldots, B_\eta$. So,

$$\mathrm{CIVPyFAAHM}^{\wp,\varrho}(B_1, B_2, B_3, \ldots\ldots, B_\eta) = \left(\frac{2}{\eta(\eta+1)}\Sigma_{i=1}^{\eta}\Sigma_{j=i}^{\eta}\left(B_i^{\wp} \otimes B_j^{\varrho}\right)\right)^{\frac{1}{\wp+\varrho}}$$

$$= \left(\frac{2}{\eta(\eta+1)}\Sigma_{i=1}^{\eta}\Sigma_{j=i}^{\eta}\left(B_i'^{\wp} \otimes B_j'^{\varrho}\right)\right)^{\frac{1}{\wp+\varrho}}$$

$$= \mathrm{CIVPyFAAHM}^{\wp,\varrho}(B_1{}', B_2{}', B_3{}', \ldots\ldots, B_\eta{}').$$

3.2. The Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Weighted Heronian Mean

In this sub-portion, the complex interval-valued Pythagorean fuzzy Aczel-Alsina weighted Heronian Mean operators are introduced, accompanied by an exposition of their fundamental properties.

Definition 9. Let $B_i = \langle[\gamma^{L}_{Bi}(x), \gamma^{U}_{Bi}(x)].e^{2\pi\iota[\theta^{L}_{Bi}(x),\theta^{U}_{Bi}(x)]}, [\omega^{L}_{Bi}(x), \omega^{U}_{Bi}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, $i = 1,2,3,\ldots\ldots\ldots,\eta$ be the family of CIVPyFNs with weight vector $w = (w_1, w_2, w_3, \ldots\ldots, w_\eta)^{\tau}$, $w_\eta \epsilon[0,1], \Sigma_{i=1}^{\eta} w_i = 1$ and $\wp > 0, \varrho > 0$. Then,

CIVPyFAAWHM is defined as follows:

$$\mathrm{CIVPyFAAWHM}^{\wp,\varrho}(B_1, B_2, B_3, \ldots\ldots.., B_\eta) = \left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}(w_iB_i)^{\wp} \otimes (w_jB_j)^{\varrho}\right)^{\frac{1}{\wp+\varrho}}.$$

Theorem 6. Let $B_i = \langle[\gamma^{L}_{Bi}(x), \gamma^{U}_{Bi}(x)].e^{2\pi\iota[\theta^{L}_{Bi}(x),\theta^{U}_{Bi}(x)]}, [\omega^{L}_{Bi}(x), \omega^{U}_{Bi}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, $i = 1,2,3,\ldots\ldots\ldots,\eta$ be the family of CIVPyFNs with weight vector $w = (w_1, w_2, w_3, \ldots\ldots, w_\eta)^{\tau}$, $w_\eta \epsilon[0,1], \Sigma_{i=1}^{\eta} w_i = 1$ and $\wp > 0, \varrho > 0$. Then, CIVPYFAAWHM is defined as follows:

$$\mathrm{CIVPyFAAWHM}^{p,q}\left(B_1, B_2, B_3, \ldots\ldots.., B_\eta\right) = \left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\ j=i}}^{\eta}\left(w_iB_i\right)^{p}\otimes\left(w_jB_j\right)^{q}\right)^{\frac{1}{p+q}} =$$

$$\Bigg\langle \left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega^{L}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\omega^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega^{U}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\omega^{U}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\ j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{U}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Proof:

$$w_iB_i = \left\langle\left[\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left[\sqrt{e^{-\left(w_i\left(-\ln\left(\omega_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_i\left(-\ln\left(\omega_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(w_i\left(-\ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_i\left(-\ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$w_jB_j=\Bigg\langle\left[\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{e^{-\left(w_j\left(-\ln\left(\omega_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_j\left(-\ln\left(\omega_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(w_j\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_j\left(-\ln\left(\theta'^{U}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

So, $(w_iB_i)^{\mathfrak{p}}$

$$=\Bigg\langle\left[\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{u}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathfrak{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Similarly,

$$(w_jB_j)^{q}=\left\langle\left[\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\omega_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\omega_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

Therefore,

$$(w_iB_i)^{p}\otimes(w_jB_j)^{q}$$

$$=\left\langle\left[\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]\right.$$

$$.e^{2\pi\iota\left[\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\omega^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\omega^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\omega^{U}_{Bj}{}^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Let $\lambda = 1 - e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}$

Then $\ln(1-\lambda) = -\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}$

utilizing this, we have $\sum_{\substack{i=1\\j=i}}^{\eta}(w_iB_i)^{p}\otimes(w_jB_j)^{q} =$

$$\Bigg\langle\left[\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta^{L}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta^{U}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta^{U}_{Bj}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}},e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}\right]$$

$$.e^{2\pi\iota\left[e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}},e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{U}_{Bj}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}\right]}\Bigg\rangle.$$

$$\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}(w_iB_i)^{p}\otimes\left(w_jB_j\right)^{q}\right)$$

$$=\Bigg\langle\left[\sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[e^{-\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}, e^{-\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}\right]$$

$$.e^{2\pi\iota\left[e^{-\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}, e^{-\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}\right]}\Bigg\rangle.$$

Hence,

$$\left(\frac{2}{n(n+1)}\sum_{\substack{i=1\\ j=i}}^{n}(w_iB_i)^{p}\otimes\left(w_jB_j\right)^{q}\right)^{\frac{1}{p+q}}=$$

$$\Bigg\langle\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(1-\left(\gamma_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(1-\left(\gamma_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(1-\left(\theta_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{n(n+1)}\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(p\left(-ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\frac{1}{ꝑ+ꝗ}\left(\frac{2}{ŋ(ŋ+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{ŋ}\left(ꝑ\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+ꝗ\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\frac{1}{ꝑ+ꝗ}\left(\frac{2}{ŋ(ŋ+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{ŋ}\left(ꝑ\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+ꝗ\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{ꝑ+ꝗ}\left(\frac{2}{ŋ(ŋ+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{ŋ}\left(ꝑ\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+ꝗ\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\frac{1}{ꝑ+ꝗ}\left(\frac{2}{ŋ(ŋ+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{ŋ}\left(ꝑ\left(-ln\left(1-e^{-\left(w_i\left(-ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+ꝗ\left(-ln\left(1-e^{-\left(w_j\left(-ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)\right)^{\frac{1}{Å}}}}\right]}\Bigg\rangle.$$

4. Complex Interval-valued Pythagorean Fuzzy Aczel-Alsina Geometric Heronian Mean Operators

In this section, based on the AA operational laws, several AOs, such as complex interval-valued Pythagorean fuzzy Aczel-Alsina geometric Heronian mean (CIVPyFAAGHM) operators, complex interval-valued Pythagorean fuzzy Aczel-Alsina weighted geometric Heronian mean (CIVPyFAAGHM) operators are presented along with their fundamental properties.

4.1. The Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Heronian Mean Operators

In this subsection, the CIVPyFAAGHM operators are introduced along with their fundamental properties.

Definition 10. Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle, i = 1,2,3, \dots \dots, ŋ$ be the family of CIVPyFNs, and $ꝑ \geq 0, ꝗ \geq 0$. Then, CIVPyFAAGHM is defined as follows:

$$CIVPyFAAGHM^{ꝑ,ꝗ}(B_1, B_2, B_3, \dots\dots, B_ŋ) = \frac{1}{ꝑ+ꝗ} \otimes_{\substack{i=1\\j=i}}^{ŋ} (ꝑB_i \oplus ꝗB_j)^{\frac{2}{ŋ(ŋ+1)}}.$$

Theorem 7. Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$,

$i = 1,2,3, \dots \dots, ŋ$ be the family of CIVPyFNs and $ꝑ \geq 0, ꝗ \geq 0$, then aggerated value of the , CIVPyFAAGHM operator is defined as follows:

$$CIVPyFAAGHM^{ꝑ,ꝗ}(B_1, B_2, B_3, \dots\dots, B_ŋ) = \frac{1}{ꝑ+ꝗ} \otimes_{\substack{i=1\\j=i}}^{ŋ} (ꝑB_i \oplus ꝗB_j)^{\frac{2}{ŋ(ŋ+1)}}$$

$$=\left\langle\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]\right.$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}$$

$$\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$\left..e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

Proof:

Consider,

$$pB_i=\left\langle\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left.\left[\sqrt{e^{-\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

$$qB_j=\left\langle\left[\sqrt{1-e^{-\left(q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(q\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\right.$$

$$\left[\sqrt{e^{-\left(q\left(-\ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(q\left(-\ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(q\left(-\ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(q\left(-\ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

So,

$$pB_i \oplus qB_j = \Bigg\langle\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$
$$\left[\sqrt{e^{-\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Therefore,

$$(pB_i \oplus qB_j)^{\frac{2}{ŋ(ŋ+1)}} = \Bigg\langle\left[\sqrt{e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$
$$\left[\sqrt{1-e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\omega_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$
$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{L}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\frac{2}{ŋ(ŋ+1)}\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^{2}\right)^{\mathring{A}}+q\left(-\ln\left(\theta'^{U}_{Bj}\right)^{2}\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$\otimes_{\substack{i=1\\j=1}}^{\eta}\left(pB_i \oplus qB_j\right)^{\frac{2}{\eta(\eta+1)}} = \left\langle \begin{array}{l} \left[\sqrt{e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{U}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{U}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\text{Å}}\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\text{Å}}\right)\right)^{\frac{1}{\text{Å}}}}}\right]}, \\ \left[\sqrt{1-e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^2\right)^{\text{Å}}+q\left(-\ln\omega_{Bj}^{L\,2}\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{1-e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\omega_{Bj}^{U}\right)^2\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right] \\ e^{2\pi\iota\left[\sqrt{1-e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\text{Å}}\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{1-e^{-\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\theta'^{U}_{Bj}\right)^2\right)^{\text{Å}}\right)\right)^{\frac{1}{\text{Å}}}}}\right]} \end{array} \right\rangle.$$

So,

$$\left(\frac{1}{p+q}\left(\otimes_{\substack{i=1\\j=1}}^{\eta}\left(pB_i \oplus qB_j\right)\right)^{\frac{2}{\eta(\eta+1)}}\right) =$$

$$\left\langle \begin{array}{l} \left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\gamma_{Bi}^{L}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\gamma_{Bj}^{L}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{L}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{L}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-\left(\theta_{Bi}^{U}\right)^2\right)\right)^{\text{Å}}+q\left(-\ln\left(1-\left(\theta_{Bj}^{U}\right)^2\right)\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right]} \\ \left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\omega_{Bi}^{L}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\omega_{Bj}^{L}\right)^2\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\omega_{Bi}^{U}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\omega_{Bj}^{U}\right)^2\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(\theta'^{L}_{Bi}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\theta'^{L}_{Bj}\right)^2\right)^{\text{Å}}\right)\right)\right)\right)^{\frac{1}{\text{Å}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln\left(\theta'^{U}_{Bi}\right)^2\right)^{\text{Å}}+q\left(-\ln\left(\theta'^{U}_{Bj}\right)^2\right)^{\text{Å}}\right)\right)\right)^{\frac{1}{\text{Å}}}}}\right]} \end{array} \right\rangle.$$

So,

$CIVPyFAAGHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_{\eta})=$

$$\left\langle \begin{array}{l} \left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-(\gamma_{Bi}^{L})^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\gamma_{Bj}^{L})^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-(\theta_{Bi}^{U})^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta_{Bj}^{U})^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-(\theta_{Bi}^{L})^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta_{Bj}^{L})^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln\left(1-(\theta_{Bi}^{U})^{2}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-(\theta_{Bj}^{U})^{2}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}, \\ \left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln(\omega_{Bi}^{L})^{2}\right)^{\mathring{A}}+q\left(-\ln(\omega_{Bj}^{L})^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln(\omega_{Bi}^{U})^{2}\right)^{\mathring{A}}+q\left(-\ln(\omega_{Bj}^{U})^{2}\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right] \\ .e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\left(p\left(-\ln(\theta'^{L}_{Bi})^{2}\right)^{\mathring{A}}+q\left(-\ln(\theta'^{L}_{Bj})^{2}\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(\frac{2}{\eta(\eta+1)}\right)\left(p\left(-\ln(\theta'^{U}_{Bi})^{2}\right)^{\mathring{A}}+q\left(-\ln(\theta'^{U}_{Bj})^{2}\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]} \end{array}\right\rangle.$$

4.2. The Complex Interval-Valued Pythagorean Fuzzy Aczel-Alsina Weighted Geometric Heronian Mean Operators

In this subsection, the CIVPyFAAWGHM operators are introduced along with their fundamental properties.

Definition 11. Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, $i = 1,2,3 \dots \dots, \eta$ be the family $of\ CIVPyF$V with weight vector $w = (w_1, w_2, w_3, \dots \dots, w_{\eta})^{\tau}$,$w_{\eta}\,\epsilon[0,1], \sum_{i=1}^{\eta} w_i = 1$and $p > 0, q > 0$. Then,

CIVPyFAAWGHM is defined as follows:

$$CIVPyFAAWGHM^{p,q}(B_1, B_2, B_3, \dots \dots .., B_{\eta}) = \frac{1}{p+q}\left(\otimes_{\substack{i=1\\j=1}}^{\eta}\left((pB_i)^{w_i} \oplus (qB_j)^{w_i}\right)^{\frac{2}{\eta(\eta+1)}}\right).$$

Theorem 8. Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, $i = 1,2,3, \dots, \eta$ be the family $of\ CIVPyF$V with weight vector $w = (w_1, w_2, w_3, \dots \dots, w_{\eta})^{\tau}$,$w_{\eta}\,\epsilon[0,1], \sum_{i=1}^{\eta} w_i = 1$and $p > 0, q > 0$. Then, CIVPYFAAWGHM is defined as follows:

$$CIVPyFAAWGHM^{p,q}(B_1, B_2, B_3, \dots \dots .., B_{\eta}) = \frac{1}{p+q}\left(\otimes_{\substack{i=1\\j=1}}^{\eta}\left(p(B_i)^{w_i} \oplus q(B_j)^{w_j}\right)\right)^{\frac{2}{\eta(\eta+1)}} =$$

$$\left\langle \left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln(\gamma_{Bi}^{L})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln(\gamma_{Bj}^{L})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln(\gamma_{Bi}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln(\gamma_{Bj}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]\right.$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln(\theta_{Bi}^{L})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j(\theta_{Bj}^{L})^{2^{\mathring{A}}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln-(\theta_{Bi}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln(\theta_{Bj}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-(\omega_{Bi}^{L})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-(\omega_{Bj}^{L})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-(\omega_{Bi}^{U})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-(\omega_{Bj}^{U})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$\left. .e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-(\theta'^{L}_{Bi})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-(\theta'^{L}_{Bj})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-(\theta'^{U}_{Bi})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-(\theta'^{U}_{Bj})^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\right\rangle.$$

Proof: Let $B_i = \langle[\gamma_{Bi}^{L}(x), \gamma_{Bi}^{U}(x)].e^{2\pi\iota[\theta_{Bi}^{L}(x),\theta_{Bi}^{U}(x)]}, [\omega_{Bi}^{L}(x), \omega_{Bi}^{U}(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x),\theta'^{U}_{Bi}(x)]}\rangle$, i=1,2,3,.........,$\eta$ be the family of CIVPyFNs and $p > 0, q > 0$, we have

$$B_i^{w_i} = \left\langle\left[\sqrt{e^{-\left(w_i\left(-\ln(\gamma_{Bi}^{L})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(w_i\left(-\ln(\gamma_{Bi}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(w_i\left(-\ln(\theta_{Bi}^{L})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(w_i\left(-\ln(\theta_{Bi}^{U})^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},\right.$$

$$\left[\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$B_j^{w_j}=\Bigg\langle\left[\sqrt{e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$\mathrm{p}(B_i)^{w_i}=\Bigg\langle\left[\sqrt{1-e^{-\left(\mathrm{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)\right)^{\mathring{A}\,\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathrm{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)\right)^{\mathring{A}\,\frac{1}{\mathring{A}}}}}\right].e^{2\pi\iota\left[\sqrt{1-e^{-\left(\mathrm{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)\right)^{\mathring{A}\,\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\mathrm{p}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{u}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)\right)^{\mathring{A}\,\frac{1}{\mathring{A}}}}}\right]},$$

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$$\left[\sqrt{e^{-\left(\text{ꝑ}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\text{ꝑ}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right].$$

$$e^{2\pi\iota\left[\sqrt{e^{-\left(\text{ꝑ}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\text{ꝑ}\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^{2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$\text{ɋ}\left(B_j\right)^{w_j}=\Bigg\langle\left[\sqrt{1-e^{-\left(\text{ɋ}\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\text{ɋ}\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(\text{ɋ}\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\text{ɋ}\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{u}\right)^{2}\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^{2}\right)\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}, \sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^{2}\right)\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}\right] . e^{2\pi\iota\left[\sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}, \sqrt{e^{-\left(q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^{2}\right)\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}\right]}\Big\rangle.$$

Therefore, $\left(p(B_i)^{w_i} \oplus q(B_j)^{w_j}\right) =$

$$\Big\langle\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)^{\AA}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)^{\AA}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}\right]$$

$$. e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)^{\AA}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{L}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}, \sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{U}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)^{\AA}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^{2}\right)^{\AA}\right)^{\frac{1}{\AA}}}\right)\right)\right)^{\AA}}}^{\frac{1}{\AA}}\right]},$$

$$\left.\left[\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\omega_{Bj}^{U\,2}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]\right.$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

Now let

$$\rho = 1-e^{-\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}.$$

$$\ln(1-\rho) = -\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}.$$

$$\otimes_{\substack{i=1\\j=i}}^{\eta}\left(p(B_i)^{w_i}\oplus q(B_j)^{w_j}\right)$$

$$=\left\langle\left[\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]\right.$$

$$.e^{2\pi\iota\left[\sqrt{e^{\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

$$\left(\otimes_{\substack{i=1\\j=i}}^{\eta}\left(p(B_i)^{w_i}\oplus q(B_j)^{w_j}\right)\right)^{\frac{2}{\eta(\eta+1)}}$$

$$=\Bigg\langle\left[\sqrt{e^{-\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{-\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{e^{\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}},\sqrt{e^{\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Big\rangle.$$

$$\frac{1}{p+q}\left(\otimes_{\substack{i=1\\j=i}}^{\eta}\left(p(B_i)^{w_i}\oplus q(B_j)^{w_j}\right)\right)^{\frac{2}{\eta(\eta+1)}} =$$

$$\Big\langle\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(\theta_{Bj}^{L}\right)^{2\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\Sigma_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln-\left(\theta_{Bi}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]},$$

$$\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}}\right]}\Bigg\rangle.$$

So, $CIVPyFAAWGHM^{p,q}(B_1, B_2, B_3, \ldots\ldots, B_\eta) =$

$$\Bigg\langle\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{L}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{L}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\gamma_{Bi}^{U}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\gamma_{Bj}^{U}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}}\right]$$

$$.\,e^{2\pi\iota\left[\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(\theta_{Bi}^{L}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(\theta_{Bj}^{L}\right)^{2^{Å}}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}},\sqrt{1-e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln-\left(\theta_{Bi}^{U}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(\theta_{Bj}^{U}\right)^2\right)^{Å}\right)^{\frac{1}{Å}}}\right)\right)^{Å}\right)\right)\right)^{\frac{1}{Å}}}}\right]},$$

$$\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{L}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\omega_{Bi}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\omega_{Bj}^{U}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]$$

$$.e^{2\pi\iota\left[\sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{L}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{L}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}, \sqrt{e^{-\left(\frac{1}{p+q}\left(\frac{2}{\eta(\eta+1)}\left(\sum_{\substack{i=1\\j=i}}^{\eta}\left(p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(1-\left(\theta'^{U}_{Bi}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(1-\left(\theta'^{U}_{Bj}\right)^2\right)\right)^{\mathring{A}}\right)^{\frac{1}{\mathring{A}}}}\right)\right)^{\mathring{A}}\right)\right)\right)\right)^{\frac{1}{\mathring{A}}}}}\right]}\Bigg\rangle.$$

5. MCDM Techniques Based on Complex Interval-valued Pythagorean Fuzzy Information

Engaging in a Multi-Criteria Decision-Making (MCDM) technique entails a systematic approach to selecting the optimal alternative from a given set, taking into account predefined attributes as specified by the decision-maker. Consider a finite set of alternatives $r = (r_1, r_2, r_3, \dots r_\eta)$, and the attributes are denoted by $w = (w_1, w_2, w_3, \dots w_\eta)$ defined by the decision-maker. In the context of CIVPyFVs, the set of finite alternatives, characterized by attributes confined within the [0,1] range, forms the closed environment. Let $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \dots \mathfrak{A}_\eta)$ be the set of weight vectors $\mathfrak{A}_i \in [0,1]$, and $\sum_{i=1}^{\eta} \mathfrak{A}_i = 1, i = (1,2,3, \dots, \eta)$. Consider a decision matrix $\mathrm{E} = \langle [\gamma_{Bi}^L(x), \gamma_{Bi}^U(x)].e^{2\pi\iota[\theta_{Bi}^L(x), \theta_{Bi}^U(x)]}, [\omega_{Bi}^L(x), \omega_{Bi}^U(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)]}\rangle_{\mathfrak{H}\times\mathring{A}}$ provide from the decision maker. Each pair $\langle [\gamma_{\mathfrak{H}\mathring{A}}^L, \gamma_{\mathfrak{H}\mathring{A}}^U].e^{2\pi\iota[\theta_{\mathfrak{H}\mathring{A}}^L, \theta_{\mathfrak{H}\mathring{A}}^U]}, [\omega_{\mathfrak{H}\mathring{A}}^L, \omega_{\mathfrak{H}\mathring{A}}^U].e^{2\pi\iota[\theta'^{L}_{\mathfrak{H}\mathring{A}}, \theta'^{U}_{\mathfrak{H}\mathring{A}}]}\rangle$ denotes the CIVPyFVs, and it must satisfy the condition of $0 \leq \left(\gamma_{Bi}^U(x)\right)^2 + \left(\omega_{Bi}^U(x)\right)^2 \leq 1$ and $0 \leq \left(\theta_{Bi}^U(x)\right)^2 + \left(\theta'^{U}_{Bi}(x)\right)^2 \leq 1$.

$$\mathrm{E} = \langle [\gamma_{Bi}^L(x), \gamma_{Bi}^U(x)].e^{2\pi\iota[\theta_{Bi}^L(x), \theta_{Bi}^U(x)]}, [\omega_{Bi}^L(x), \omega_{Bi}^U(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)]}\rangle_{\mathfrak{H}\times\mathring{A}}$$

$$= \begin{bmatrix} \langle[\gamma_{11}^L,\gamma_{11}^U].e^{2\pi\iota[\theta_{11}^L,\theta_{11}^U]},[\omega_{11}^L,\omega_{11}^U].e^{2\pi\iota[\theta'^{L}_{11},\theta'^{U}_{11}]}\rangle & \langle[\gamma_{12}^L,\gamma_{12}^U].e^{2\pi\iota[\theta_{12}^L,\theta_{12}^U]},[\omega_{12}^L,\omega_{12}^U].e^{2\pi\iota[\theta'^{L}_{12},\theta'^{U}_{12}]}\rangle & \cdots & \langle[\gamma_{1\mathring{A}}^L,\gamma_{1\mathring{A}}^U].e^{2\pi\iota[\theta_{1\mathring{A}}^L,\theta_{1\mathring{A}}^U]},[\omega_{1\mathring{A}}^L,\omega_{1\mathring{A}}^U].e^{2\pi\iota[\theta'^{L}_{1\mathring{A}},\theta'^{U}_{1\mathring{A}}]}\rangle \\ \langle[\gamma_{21}^L,\gamma_{21}^U].e^{2\pi\iota[\theta_{21}^L,\theta_{21}^U]},[\omega_{21}^L,\omega_{21}^U].e^{2\pi\iota[\theta'^{L}_{21},\theta'^{U}_{21}]}\rangle & \langle[\gamma_{22}^L,\gamma_{22}^U].e^{2\pi\iota[\theta_{22}^L,\theta_{22}^U]},[\omega_{22}^L,\omega_{22}^U].e^{2\pi\iota[\theta'^{L}_{22},\theta'^{U}_{22}]}\rangle & \cdots & \langle[\gamma_{2\mathring{A}}^L,\gamma_{2\mathring{A}}^U].e^{2\pi\iota[\theta_{2\mathring{A}}^L,\theta_{2\mathring{A}}^U]},[\omega_{2\mathring{A}}^L,\omega_{2\mathring{A}}^U].e^{2\pi\iota[\theta'^{L}_{2\mathring{A}},\theta'^{U}_{2\mathring{A}}]}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle[\gamma_{\mathfrak{H}1}^L,\gamma_{\mathfrak{H}1}^U].e^{2\pi\iota[\theta_{\mathfrak{H}1}^L,\theta_{\mathfrak{H}1}^U]},[\omega_{\mathfrak{H}1}^L,\omega_{\mathfrak{H}1}^U].e^{2\pi\iota[\theta'^{L}_{\mathfrak{H}1},\theta'^{U}_{\mathfrak{H}1}]}\rangle & \langle[\gamma_{\mathfrak{H}2}^L,\gamma_{\mathfrak{H}2}^U].e^{2\pi\iota[\theta_{\mathfrak{H}2}^L,\theta_{\mathfrak{H}2}^U]},[\omega_{\mathfrak{H}2}^L,\omega_{\mathfrak{H}2}^U].e^{2\pi\iota[\theta'^{L}_{\mathfrak{H}2},\theta'^{U}_{\mathfrak{H}2}]}\rangle & \cdots & \langle[\gamma_{\mathfrak{H}\mathring{A}}^L,\gamma_{\mathfrak{H}\mathring{A}}^U].e^{2\pi\iota[\theta_{\mathfrak{H}\mathring{A}}^L,\theta_{\mathfrak{H}\mathring{A}}^U]},[\omega_{\mathfrak{H}\mathring{A}}^L,\omega_{\mathfrak{H}\mathring{A}}^U].e^{2\pi\iota[\theta'^{L}_{\mathfrak{H}\mathring{A}},\theta'^{U}_{\mathfrak{H}\mathring{A}}]}\rangle \end{bmatrix}.$$

We demonstrate the identification of furthermost appropriate alternative by employing the AOs of CIVPyFAAWHM also the CIVPyFAAWGHM operators within the framework of MCDM under the context of CIVPyF information. The assessment of Complex Interval-valued fuzzy information within a (MCDM) technique involves a systematic evaluation through the algorithmic steps outlined.

Step 1: The decision-maker gathers information in the form of (CIVPyFVs) and presents the entire set of provided information through decision matrices.

Step 2: It is imperative to convert the given decision matrix $\mathrm{E} = \langle [\gamma_{Bi}^L(x), \gamma_{Bi}^U(x)].e^{2\pi\iota[\theta_{Bi}^L(x), \theta_{Bi}^U(x)]}, [\omega_{Bi}^L(x), \omega_{Bi}^U(x)].e^{2\pi\iota[\theta'^{L}_{Bi}(x), \theta'^{U}_{Bi}(x)]}\rangle_{\mathfrak{H}\times\mathring{A}}$ into the normalization matrix $\bar{\mathrm{E}} =$

$\langle[\gamma^L_{B\iota}(x), \gamma^U_{B\iota}(x)].e^{2\pi\iota[\theta^L_{B\iota}(x), \theta^U_{B\iota}(x)]}, [\omega^L_{B\iota}(x), \omega^U_{B\iota}(x)].e^{2\pi\iota[\theta'^L_{B\iota}(x), \theta'^U_{B\iota}(x)]}\rangle_{\mathfrak{H}\times Å}$ if the set of attributes includes diverse types, such as benefit types and cost types. If the attributes within the set are uniform, there is no requirement to undergo the transformation of the decision matrix into a normalization matrix.

Step 3: We apply our tough proposed methodologies, including (CIVPyFAAWHM) and (CIVPyFAAWGHM) operators. In the decision-making process, these methodologies are applied to identify the optimal option from a finite set of alternatives, represented as $r = (r_1, r_2, r_3, \dots r_\eta)$, taking into consideration specific characteristics defined for the decision-making process.

Step 4: Calculate score values by employing the derived outcomes from (CIVPyFAAWHM) and (CIVPyFAAWGHM) operators.

$$\mathcal{S}(B) = \frac{1}{2}\langle(\gamma^L(x))^2 + (\gamma^U(x))^2 + (\theta^L(x))^2 + (\theta^U(x))^2 - (\omega^L(x))^2 - (\omega^U(x))^2 - (\theta'^L(x))^2 - (\theta'^U(x))^2\rangle \in [-1,1]$$

Step 5: Upon calculating the score values, the next step involves investigating a more suitable alternative by rearranging the score values through a systematic ranking and ordering process.

6. Numerical Example

In the ongoing examination of strategies aimed at improving smart city logistics, the assessment process is being conducted with a focus on various criteria and considerations. They are currently identifying five evaluation criteria and considering five different strategies for the ongoing assessment. The assessment encompasses specific criteria: impact on environmental sensitivity $ŧ_1$, effect on logistics costs $ŧ_2$, impact on time delivery $ŧ_3$, the effect of logistic service quality $ŧ_4$, impact on speed and traffic density in transport modes $ŧ_5$. The analogous weight vectors is $\omega = (0.31, 0.19, 0.23, 0.16, 0.09)$.

6.1. The Estimate Method of MCDM Technique

Step 1. Decision makers evaluate assessment (ψ = 1, 2, 3, 4, 5) with respect to the criteria's (ŧ= 1, 2, 3, 4, 5) by using CIVPyFNs and obtain a decision matrix as shown in Table 1.

Table 1: Decision Matrix.

	$ŧ_1$	$ŧ_2$	$ŧ_3$	$ŧ_4$	$ŧ_5$
ψ_1	$([(0.06),(0.01)]e^{2\pi\iota[(0.06),(0.40)]}$ $[(0.28),(0.27)]e^{2\pi\iota[(0.12),(0.03)]})$	$([(0.06),(0.29)]e^{2\pi\iota[(0.06),(0.44)]}$ $[(0.67),(0.07)]e^{2\pi\iota[(0.12),(0.16)]})$	$([(0.32),(0.11)]e^{2\pi\iota[(0.41),(0.16)]}$ $[(0.06),(0.03)]e^{2\pi\iota[(0.07),(0.06)]})$	$([(0.17),(0.09)]e^{2\pi\iota[(0.56),(0.29)]}$ $[(0.01),(0.38)]e^{2\pi\iota[(0.13),(0.04)]})$	$([(0.21),(0.19)]e^{2\pi\iota[(0.13),(0.04)]}$ $[(0.06),(0.04)]e^{2\pi\iota[(0.26),(0.45)]})$
ψ_2	$([(0.21),(0.19)]e^{2\pi\iota[(0.33),(0.27)]}$ $[(0.05),(0.09)]e^{2\pi\iota[(0.03),(0.02)]})$	$([(0.01),(0.38)]e^{2\pi\iota[(0.17),(0.33)]}$ $[(0.38),(0.15)]e^{2\pi\iota[(0.08),(0.09)]})$	$([(0.18),(0.12)]e^{2\pi\iota[(0.05),(0.24)]}$ $[(0.01),(0.25)]e^{2\pi\iota[(0.34),(0.15)]})$	$([(0.13),(0.04)]e^{2\pi\iota[(0.56),(0.29)]}$ $[(0.06),(0.45)]e^{2\pi\iota[(0.65),(0.29)]})$	$([(0.38),(0.15)]e^{2\pi\iota[(0.16),(0.40)]}$ $[(0.16),(0.03)]e^{2\pi\iota[(0.21),(0.19)]})$
ψ_3	$([(0.46),(0.11)]e^{2\pi\iota[(0.39),(0.02)]}$ $[(0.07),(0.27)]e^{2\pi\iota[(0.05),(0.29)]})$	$([(0.16),(0.40)]e^{2\pi\iota[(0.18),(0.23)]}$ $[(0.22),(0.12)]e^{2\pi\iota[(0.05),(0.18)]})$	$([(0.41),(0.23)]e^{2\pi\iota[(0.24),(0.32)]}$ $[(0.03),(0.08)]e^{2\pi\iota[(0.26),(0.01)]})$	$([(0.23),(0.32)]e^{2\pi\iota[(0.67),(0.09)]}$ $[(0.16),(0.01)]e^{2\pi\iota[(0.58),(0.79)]})$	$([(0.25),(0.09)]e^{2\pi\iota[(0.56),(0.29)]}$ $[(0.12),(0.38)]e^{2\pi\iota[(0.01),(0.38)]})$
ψ_4	$([(0.36),(0.23)]e^{2\pi\iota[(0.17),(0.33)]}$ $[(0.13),(0.04)]e^{2\pi\iota[(0.12),(0.18)]})$	$([(0.19),(0.25)]e^{2\pi\iota[(0.15),(0.13)]}$ $[(0.08),(0.07)]e^{2\pi\iota[(0.36),(0.11)]})$	$([(0.24),(0.18)]e^{2\pi\iota[(0.07),(0.15)]}$ $[(0.16),(0.45)]e^{2\pi\iota[(0.67),(0.16)]})$	$([(0.21),(0.19)]e^{2\pi\iota[(0.25),(0.09)]}$ $[(0.06),(0.29)]e^{2\pi\iota[(0.62),(0.38)]})$	$([(0.22),(0.16)]e^{2\pi\iota[(0.21),(0.12)]}$ $[(0.06),(0.40)]e^{2\pi\iota[(0.03),(0.32)]})$
ψ_5	$([(0.28),(0.19)]e^{2\pi\iota[(0.36),(0.12)]}$ $[(0.03),(0.28)]e^{2\pi\iota[(0.21),(0.32)]})$	$([(0.22),(0.56)]e^{2\pi\iota[(0.39),(0.21)]}$ $[(0.07),(0.09)]e^{2\pi\iota[(0.12),(0.09)]})$	$([(0.23),(0.27)]e^{2\pi\iota[(0.46),(0.34)]}$ $[(0.45),(0.29)]e^{2\pi\iota[(0.06),(0.04)]})$	$([(0.21),(0.32)]e^{2\pi\iota[(0.56),(0.03)]}$ $[(0.06),(0.04)]e^{2\pi\iota[(0.38),(0.15)]})$	$([(0.06),(0.09)]e^{2\pi\iota[(0.16),(0.14)]}$ $[(0.26),(0.29)]e^{2\pi\iota[(0.35),(0.29)]})$

Step 2. In the analysis, we distinguish between cost-type and benefit-type criteria, where, $ŧ_i$ = 2, 3, 4 are considered benefit type, while $ŧ_i$ = 1,5 are categorized as cost type. The decision matrix has been normalized, as illustrated in Table 2.

Table 2: Normalized Decision Matrix.

	$ŧ_1$	$ŧ_2$	$ŧ_3$	$ŧ_4$	$ŧ_5$
ψ_1	$([(0.28),(0.27)]e^{2\pi\iota[(0.12),(0.03)]}$ $[(0.06),(0.11)]e^{2\pi\iota[(0.06),(0.40)]})$	$([(0.06),(0.29)]e^{2\pi\iota[(0.06),(0.44)]}$ $[(0.67),(0.07)]e^{2\pi\iota[(0.12),(0.16)]})$	$([(0.32),(0.11)]e^{2\pi\iota[(0.41),(0.16)]}$ $[(0.06),(0.03)]e^{2\pi\iota[(0.07),(0.06)]})$	$([(0.17),(0.09)]e^{2\pi\iota[(0.56),(0.29)]}$ $[(0.01),(0.38)]e^{2\pi\iota[(0.13),(0.04)]})$	$([(0.06),(0.04)]e^{2\pi\iota[(0.26),(0.45)]}$ $[(0.21),(0.19)]e^{2\pi\iota[(0.13),(0.04)]})$
ψ_2	$([(0.05),(0.09)]e^{2\pi\iota[(0.03),(0.02)]}$ $[(0.21),(0.19)]e^{2\pi\iota[(0.33),(0.27)]})$	$([(0.01),(0.38)]e^{2\pi\iota[(0.17),(0.33)]}$ $[(0.38),(0.15)]e^{2\pi\iota[(0.08),(0.09)]})$	$([(0.18),(0.12)]e^{2\pi\iota[(0.05),(0.24)]}$ $[(0.01),(0.25)]e^{2\pi\iota[(0.34),(0.15)]})$	$([(0.13),(0.04)]e^{2\pi\iota[(0.56),(0.29)]}$ $[(0.06),(0.45)]e^{2\pi\iota[(0.05),(0.29)]})$	$([(0.16),(0.03)]e^{2\pi\iota[(0.21),(0.19)]}$ $[(0.38),(0.15)]e^{2\pi\iota[(0.16),(0.40)]})$
ψ_3	$([(0.07),(0.27)]e^{2\pi\iota[(0.05),(0.29)]}$ $[(0.46),(0.11)]e^{2\pi\iota[(0.39),(0.02)]})$	$([(0.16),(0.40)]e^{2\pi\iota[(0.18),(0.23)]}$ $[(0.22),(0.12)]e^{2\pi\iota[(0.05),(0.18)]})$	$([(0.41),(0.23)]e^{2\pi\iota[(0.24),(0.32)]}$ $[(0.03),(0.08)]e^{2\pi\iota[(0.26),(0.01)]})$	$([(0.23),(0.32)]e^{2\pi\iota[(0.67),(0.09)]}$ $[(0.16),(0.01)]e^{2\pi\iota[(0.08),(0.19)]})$	$([(0.12),(0.38)]e^{2\pi\iota[(0.01),(0.38)]}$ $[(0.25),(0.09)]e^{2\pi\iota[(0.56),(0.29)]})$
ψ_4	$([(0.13),(0.04)]e^{2\pi\iota[(0.22),(0.18)]}$ $[(0.36),(0.23)]e^{2\pi\iota[(0.17),(0.33)]})$	$([(0.19),(0.25)]e^{2\pi\iota[(0.15),(0.13)]}$ $[(0.08),(0.07)]e^{2\pi\iota[(0.36),(0.11)]})$	$([(0.24),(0.18)]e^{2\pi\iota[(0.07),(0.15)]}$ $[(0.16),(0.45)]e^{2\pi\iota[(0.67),(0.16)]})$	$([(0.21),(0.19)]e^{2\pi\iota[(0.25),(0.09)]}$ $[(0.06),(0.29)]e^{2\pi\iota[(0.02),(0.38)]})$	$([(0.06),(0.40)]e^{2\pi\iota[(0.03),(0.32)]}$ $[(0.22),(0.16)]e^{2\pi\iota[(0.21),(0.12)]})$
ψ_5	$([(0.03),(0.28)]e^{2\pi\iota[(0.21),(0.32)]}$ $[(0.28),(0.19)]e^{2\pi\iota[(0.36),(0.12)]})$	$([(0.22),(0.56)]e^{2\pi\iota[(0.39),(0.21)]}$ $[(0.07),(0.09)]e^{2\pi\iota[(0.12),(0.09)]})$	$([(0.23),(0.27)]e^{2\pi\iota[(0.46),(0.34)]}$ $[(0.45),(0.29)]e^{2\pi\iota[(0.06),(0.04)]})$	$([(0.21),(0.32)]e^{2\pi\iota[(0.56),(0.03)]}$ $[(0.06),(0.04)]e^{2\pi\iota[(0.08),(0.15)]})$	$([(0.26),(0.29)]e^{2\pi\iota[(0.35),(0.29)]}$ $[(0.06),(0.09)]e^{2\pi\iota[(0.16),(0.14)]})$

Step 3. We smeared our advised aggregation operators on normalized decision matrix Table 3 for $\mathring{A}$ =10. All attained results are portrayed in Table 3.

Table 3: Consequence of the CIVPyFAAWHM and CIVPyFAAWGHM operators at $\mathring{A} = 10$.

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$
CIVPyFAAWHM	$\left([0.04273,0.05757]e^{2\pi\iota[0.0314,0.0455]}, [0.4774,0.2759]e^{2\pi\iota[0.43007,0.4103]}\right)$	$\left([0.019,0.2739]e^{2\pi\iota[0.0894,0.1568]}, [0.1846,0.6563]e^{2\pi\iota[0.3984,0.2107]}\right)$	$\left([0.204,0.1313]e^{2\pi\iota[0.0682,0.1686]}, [0.5017,0.4815]e^{2\pi\iota[0.6789,0.2032]}\right)$	$\left([0.1554,0.0607]e^{2\pi\iota[0.2819,0.0429]}, [0.1899,0.4726]e^{2\pi\iota[0.1494,0.4171]}\right)$	$\left([0.0715,0.0439]e^{2\pi\iota[0.01807,0.2252]}, [0.4081,0.2137]e^{2\pi\iota[.5682,0.4266]}\right)$
CIVPyFAAWGHM	$\left([0.2962,0.3446]e^{2\pi\iota[0.2784,0.3532]}, [0.0905,0.1253]e^{2\pi\iota[0.0902,0.0335]}\right)$	$\left([0.2839,0.59901]e^{2\pi\iota[0.4473,0.4453]}, [0.0771,0.0817]e^{2\pi\iota[0.06901,0.0949]}\right)$	$\left([0.4311,0.3338]e^{2\pi\iota[0.5109,0.4016]}, [0.0185,0.0185]e^{2\pi\iota[0.05002,0.0182]}\right)$	$\left([0.2794,0.3840]e^{2\pi\iota[0.6701,0.3275]}, [0.02,0.0.0183]e^{2\pi\iota[0.0312,0.0643]}\right)$	$\left([0.3239,0.439]e^{2\pi\iota[0.4102,0.4558]}, [0.079,0.0978]e^{2\pi\iota[0.1494,0.0638]}\right)$

Step 4. By applying Definition 2, we calculated the score values which are shown in Table 4, where the graphical result of the proposed operators CIVPyFAAWHM and CIVPyFAAWGHM are presented in Fig. 1.

Step 5. To determine the most suitable alternative, we conduct ranking and ordering.

Table 4: Score values of the CIVPyFAAWHM and CIVPyFAAWGHM operators.

	$\mathcal{SB}_1$	$\mathcal{SB}_2$	$\mathcal{SB}_3$	$\mathcal{SB}_4$	$\mathcal{SB}_5$	Ranking and ordering
CIVPyFAAWHM	-0.32463	-0.28006	-0.44698	-0.17334	-0.32950	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
CIVPyFAAWGHM	0.18787	0.40573	0.35557	0.38811	0.31581	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$

6.2. Influence of the Parameters $\mathring{A}$, ꝑ and ꝗ on Final Ranking Order

In this sub-portion, the variation of the influence $\mathring{A}$, ꝑ and ꝗ are investigated. From Table 5 and Fig. 2, one can observe that when different values to the parameter $\mathring{A}$ assigned, different score values are obtained, while the ranking order remains the same utilizing CIVPyFAAWHM operators.

Observing Table 6 and Fig. 3, it becomes apparent that assigning various values to the parameters ꝑ and ꝗ leads to different score values, yet the ranking order remains consistent when utilizing CIVPyFAAWHM operators.

From Table 7 and Fig. 4, it is evident that varying the parameter $\mathring{A}$ yields different score values when employing the CIVPyFAAWGHM operators. Specifically, as depicted in Table 7 and Fig. 4, an increase in the values of the parameter Å results in a decrease in the obtained score values when utilizing the CIVPyFAAWGHM operator. This flexibility allows experts to tailor the values of the parameter Å to meet their specific requirements when tackling practical MCDM/MCGDM problems.

From Table 8 and Fig. 5, it becomes evident that altering the values assigned to the parameters ᵽ and ꝗ results in varying score values, while the ranking order remains consistent when employing CIVPyFAAWGHM operators. In real-world scenarios, experts determine the values of the three parameters based on the specific demands of the circumstances at hand.

Table 5: Consequence of the score values of the CIVPyFAAWHM by the variation of Å.

	$\mathcal{SB}_1$	$\mathcal{SB}_2$	$\mathcal{SB}_3$	$\mathcal{SB}_4$	$\mathcal{SB}_5$	Ranking and ordering
Å = 10	-0.32464	-0.28006	-0.44699	-0.17334	-0.32951	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 13	-0.31357	-0.27845	-0.43672	-0.16872	-0.32407	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 25	-0.29898	-0.27668	-0.42098	-0.1617	-0.31607	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 30	-0.29677	-0.27642	-0.41826	-0.16043	-0.3146	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 45	-0.29329	-0.27602	-0.42453	-0.15828	-0.31207	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 54	-0.29216	-0.27589	-0.4123	-0.15754	-0.31121	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 63	-0.29134	-0.27579	-0.40966	-0.15702	-0.31058	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
Å = 75	-0.29056	-0.27569	-0.42067	-0.1565	-0.30998	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$

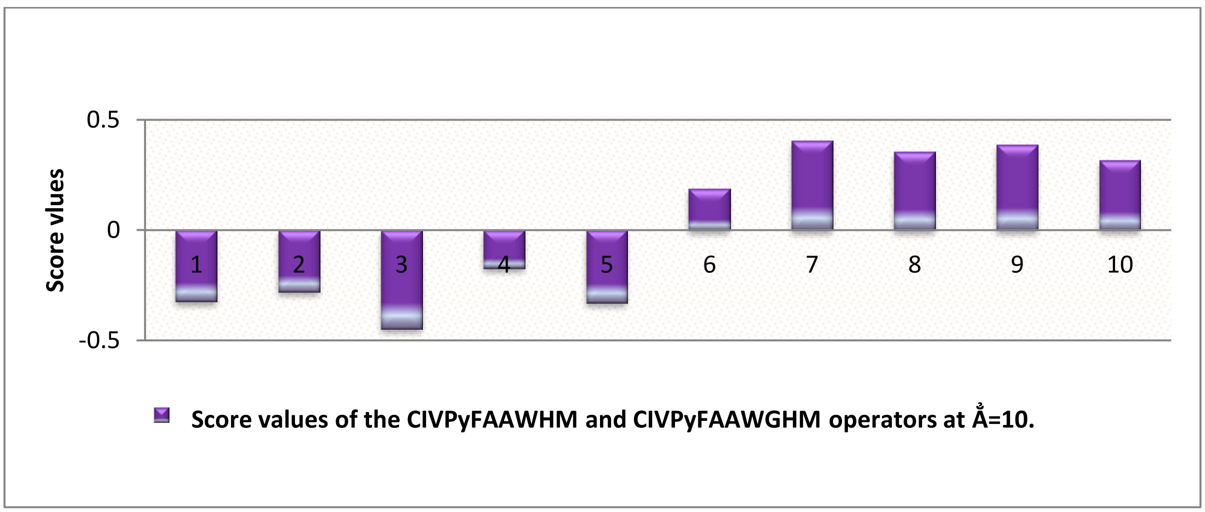

Figure 1: Graphically representation of CIVPyFAAWHM and CIVPyFAAWGHM operators at Å=10.

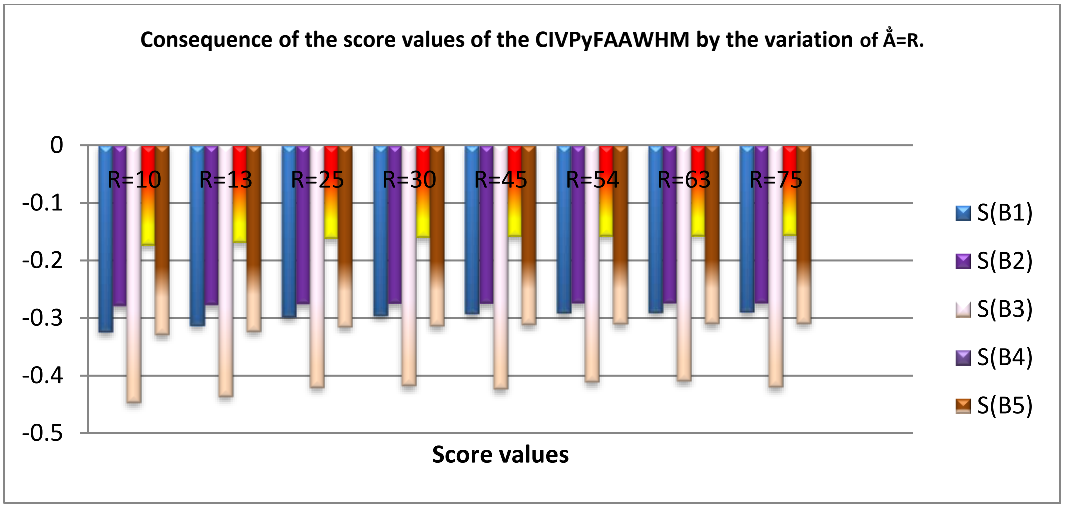

Figure 2: Graphically representation of CIVPyFAAWHM by the variation of Å=10.

Table 6: Consequence of the score values of the CIVPyFAAWHM by the variation of ꝑ and ꝗ.

	$\mathcal{SB}_1$	$\mathcal{SB}_2$	$\mathcal{SB}_3$	$\mathcal{SB}_4$	$\mathcal{SB}_5$	Ranking and ordering
ꝑ=3, ꝗ =5	-0.29329	-0.27602	-0.42453	-0.15828	-0.31207	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
ꝑ=1, ꝗ =5	-0.29298	-0.27624	-0.42499	-0.15822	-0.31152	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
ꝑ=2, ꝗ =4	-0.29204	-0.27585	-0.4247	-0.15826	-0.3105	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
ꝑ=4, ꝗ =2	-0.29315	-0.27674	-0.42301	-0.1582	-0.31174	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
ꝑ=5, ꝗ =1	-0.29331	-0.27685	-0.42174	-0.15802	-0.31226	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
ꝑ=15, ꝗ =12	-0.29362	-0.27655	-0.42367	-0.15826	-0.31136	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$

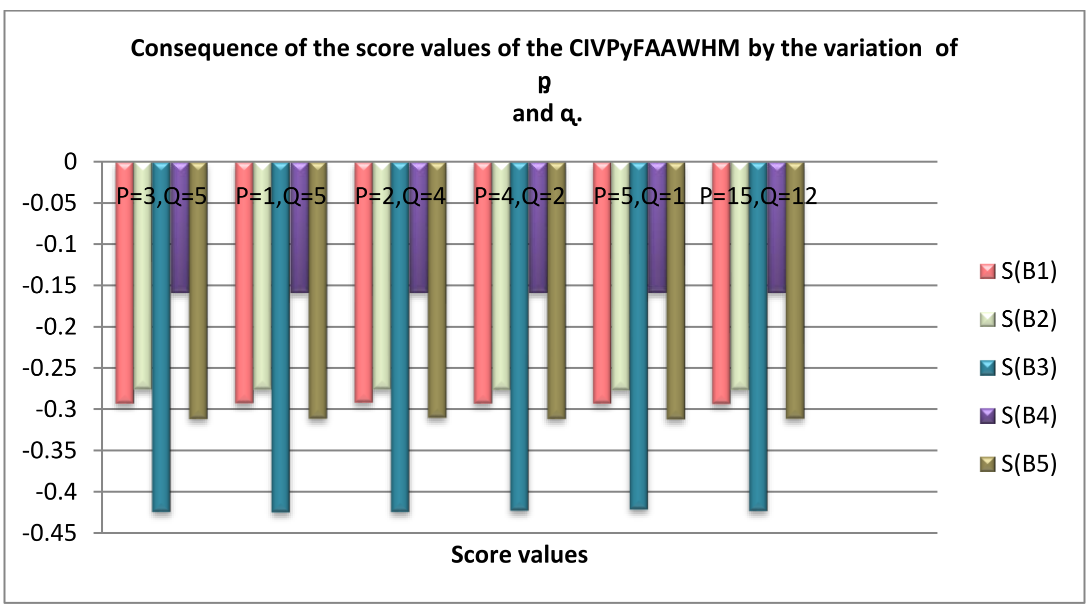

Figure 3: Graphically representation of CIVPyFAAWHM by the variation of ꝑ and ꝗ.

Table 7: Consequence of the score values of the CIVPyFAAWGHM by the variation of Å.

	$\mathcal{SB}_1$	$\mathcal{SB}_2$	$\mathcal{SB}_3$	$\mathcal{SB}_4$	$\mathcal{SB}_5$	Ranking and ordering
Å = 10	0.187875	0.405731	0.355574	0.388111	0.315817	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 13	0.174685	0.391446	0.338621	0.375959	0.302074	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 25	0.154534	0.368533	0.311352	0.35889	0.281563	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 30	0.151057	0.364386	0.306985	0.356126	0.278057	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 45	0.145445	0.357478	0.298188	0.351711	0.272361	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 54	0.143633	0.355176	0.295447	0.350265	0.270495	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 63	0.142353	0.353533	0.293489	0.349234	0.269168	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
Å = 75	0.141134	0.351955	0.291611	0.348244	0.267896	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$

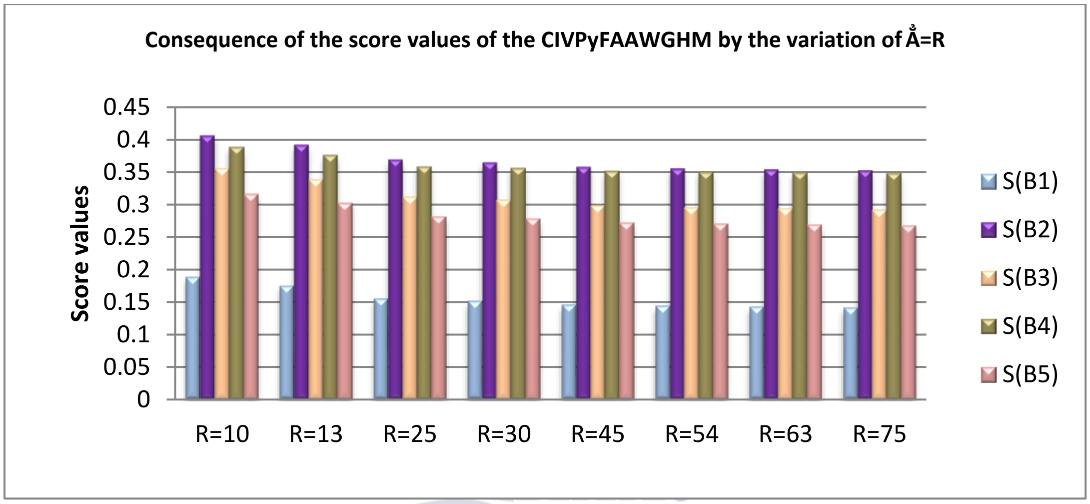

Figure 4: Graphically representation of CIVPyFAAWGHM by the varation of Å.

Table 8: Consequence of the score values of the CIVPyFAAWGHM by the varation of ꝑ and ꝗ.

	$\mathcal{SB}_1$	$\mathcal{SB}_2$	$\mathcal{SB}_3$	$\mathcal{SB}_4$	$\mathcal{SB}_5$	Ranking and ordering
ꝑ=3, ꝗ =5	0.14536	0.357791	0.298778	0.35171	0.272183	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
ꝑ=1, ꝗ =5	0.145377	0.357604	0.298394	0.351715	0.272324	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
ꝑ=2, ꝗ =4	0.145394	0.355968	0.296379	0.351558	0.272085	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
ꝑ=4, ꝗ =2	0.1452	0.354437	0.294875	0.349817	0.280409	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
ꝑ=5, ꝗ =1	0.144772	0.356684	0.297159	0.351643	0.272298	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
ꝑ=15, ꝗ =12	0.145445	0.357478	0.298188	0.351711	0.272361	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$

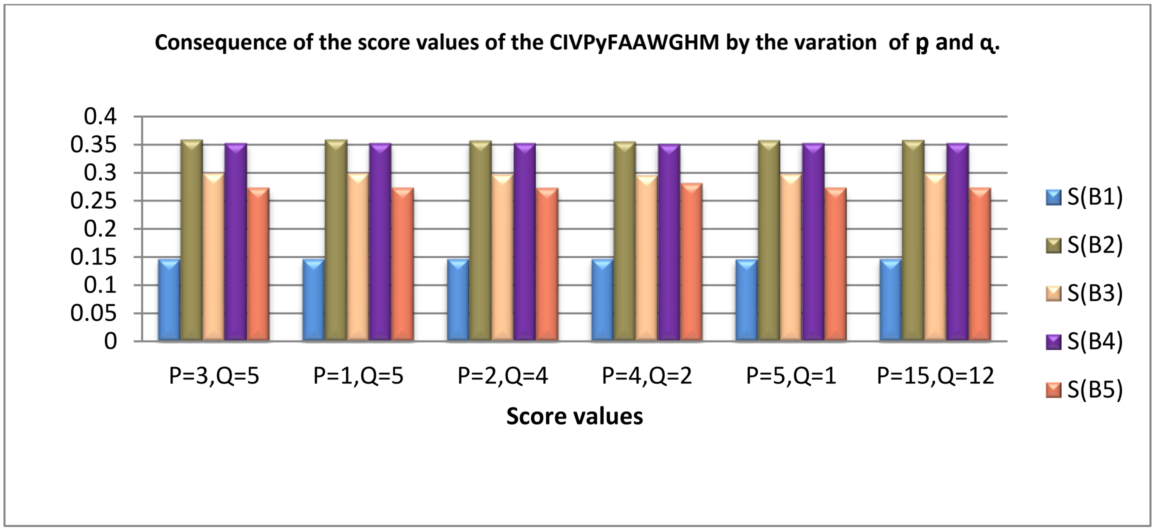

Figure 5: Graphically representation of CIVPyFAAWGHM by the varation of ꝑ and ꝗ.

7. Comparison Analysis

In this section, a comprehensive comparison is conducted between our proposed aggregation operators (AOs) and existing aggregation operators. The analysis encompasses not only a detailed examination of the advantages offered by our AOs but also an acknowledgment of their limitations. To facilitate a clear understanding of the comparative evaluation, we present the findings in a structured table below, providing a visual representation of the strengths and potential constraints associated with each aggregation operator. This comparative analysis aims to offer insights into the effectiveness and applicability of our proposed AOs within the context of aggregation operations.

Table 9: Comparative analysis of the proposed operators with existing operators.

Aggregation Operators	Scores values	Ranking
CIVPyFWA [44]	$\mathcal{SB}_1 = -0.36844, \mathcal{SB}_2 = -0.35718, \mathcal{SB}_3 = -0.32846, \mathcal{SB}_4 = -0.32932, \mathcal{SB}_5 = -0.36997$	$\mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5$
CIVPyFWG [44]	$\mathcal{SB}_1 = 0.343669, \mathcal{SB}_2 = 0.369155, \mathcal{SB}_3 = 0.379503, \mathcal{SB}_4 = 0.380674, \mathcal{SB}_5 = 0.354228$	$\mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_2 > \mathcal{SB}_5 > \mathcal{SB}_1$
IVCPyFEWA [45]	$\mathcal{SB}_1 = -0.06127, \mathcal{SB}_2 = -0.05413, \mathcal{SB}_3 = -0.02453, \mathcal{SB}_4 = -0.01057, \mathcal{SB}_5 = -0.06639$	$\mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5$
IVCPyFEWG [45]	$\mathcal{SB}_1 = 0.028944, \mathcal{SB}_2 = 0.107278, \mathcal{SB}_3 = 0.094756, \mathcal{SB}_4 = 0.171844, \mathcal{SB}_5 = 0.071468$	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
CIVPyFEWG [51]	$\mathcal{SB}_1 = 0.028944, \mathcal{SB}_2 = 0.107278, \mathcal{SB}_3 = 0.094756, \mathcal{SB}_4 = 0.171844, \mathcal{SB}_5 = 0.071468$	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$
CIVPyFAAWHM	$\mathcal{SB}_1 = -0.32464, \mathcal{SB}_2 = -0.28006, \mathcal{SB}_3 = -0.44699, \mathcal{SB}_4 = -0.17334, \mathcal{SB}_5 = -0.32951$	$\mathcal{SB}_4 > \mathcal{SB}_2 > \mathcal{SB}_1 > \mathcal{SB}_5 > \mathcal{SB}_3$
CIVPyFAAWGHM	$\mathcal{SB}_1 = 0.187875, \mathcal{SB}_2 = 0.405731, \mathcal{SB}_3 = 0.355574, \mathcal{SB}_4 = 0.388111, \mathcal{SB}_5 = 0.315817$	$\mathcal{SB}_2 > \mathcal{SB}_4 > \mathcal{SB}_3 > \mathcal{SB}_5 > \mathcal{SB}_1$

The proposed AOs stand out due to several distinctive advantages that set them apart from existing counterparts. Unlike conventional AOs, which often consider one feature at a time, our AOs exhibit the unique ability to simultaneously account for the correlation between input data. This dual functionality enables the removal of undesirable data influences from the final ranking result, enhancing the robustness and precision of the aggregation process. Additionally, the proposed AOs are grounded in the Aczel-Alsina operating law, characterized by generalized parameters. This foundational basis contributes to the practicality of the proposed MCDM model, particularly in addressing complex real-world problems within the challenging context of CIVPyF environments. From Table 9 and Fig. 6, we analyzed that the most suitable alternative is $\mathcal{SB}_4$ by using the existing AOs say CIVPyFWA [44], CIVPyFWG [44], IVCPyFEWA [45] and proposed AOs CIVPyFAAWHM. By using the operators IVCPyFEWG [45], CIVPyFEWG [51], and CIVPyFAAWHM, we deduce that $\mathcal{SB}_2$ is the most suitable alternative.

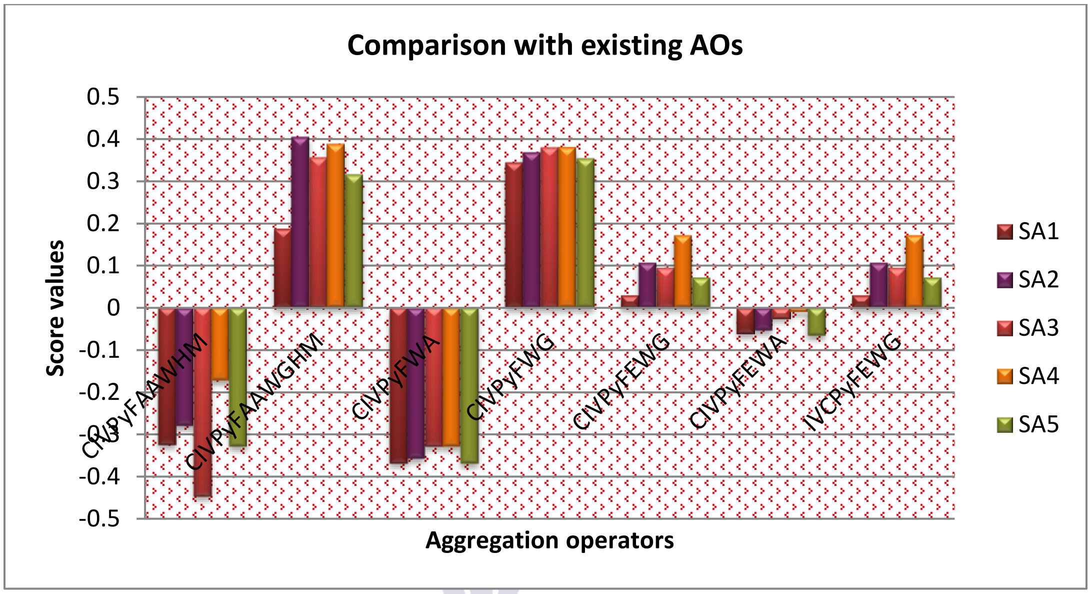

Figure 6: Graphically representation of comparison with existing AOs.

8. Conclusion

The MCDM technique, a subfield of operations research, systematically evaluates various options to find the optimal solution by comparing potential answers. In diverse fields like social science, economics, and medical science, MCDM experiments are frequently confronted and widely applied. The main objective of this article is to address uncertainty and ambiguity within the CIVPyF information system. Our study delved into the notions of triangular norms and their simplification, exploring robust aggregation tools such as Aczel Alsina operations. Our exploration focused on the schemes of HM and GHM tools, aiming to address unsatisfactory info also establish robust interrelationships among different arguments. By employing the Aczel Alsina operations within the framework of CIVPyF information, specifically, the CIVPyFAAHM and CIVPyFAAWHM operators, we compiled a list of specific aggregation operators (AOs) for HM tools.

Furthermore, we stretched the principle of GHM implements using AA operations and introduced a novel set of AOs known as CIVPyFAAGHM and CIVPyFAAWGHM operators. The particular goods of these innovative methodologies are also offered. We applied our devised scheme, incorporating the CIVPyFAAWHM and CIVPyFAAWGHM operators, to select best selections within the framework of the MCDM technique. To assess the effectiveness of the proposed model, an example case is presented, and a comparative study is conducted to validate and substantiate the proposed methodologies. A notable drawback of the method is its inability to manage situations that encompass a wide range of decision information types. Our intention is to extend the scope of the first aggregation operator to encompass various fuzzy structures, such as cubic IVPy fuzzy sets and neutrosophic sets.

Additionally, we will employ a newly developed algorithm to tackle more challenging decision-making situations [46] , [47] , [48].

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