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A COMPARATIVE STUDY OF REPEATED AG-GROUPOIDS

M. Yousaf, M. Rashad, I. Ahmad, Wajahat Ali Khan
Department of Mathematics, University of Malakand, Pakistan.

musaf2004@yahoo.com , rashad@uom.edu.pk, iahmaad@hotmail.com, wajahat@uom.edu.pk
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Corresponding Author: Wajahat Ali Khan
wajahat@uom.edu.pk*

Abstract

A magma that satisfies the left invertive law: $ab \cdot c = cb \cdot a$ is called an AG-groupoid. The concept of "repeated LA-semigroup" that arise in various papers, which satisfies the two identities: $(t \cdot u)(v \cdot w) = (uu) \cdot (vv)$, $\forall t, u, v, w$ (Inner Repeated), $(t \cdot u)(v \cdot w) = (tt) \cdot (ww)$, $\forall t, u, v, w$ (Outer Repeated) is investigated. The enumerations for these subclasses up to order 6 is provided using the modern computational techniques of GAP. Furthermore, various relations of these subclasses are investigated with some other existing subclasses of AG-groupoids and with other relevant algebraic structures. Various examples and counterexamples are produced with Prover-9 and Mace-4 to strengthen the validity of the produced results.

INTRODUCTION

An LA-semigroup J is a groupoid that satisfies the "left invertive law: $(kl)m = (ml)k$, for all , for all k, l, m" in J and generalizes a commutative semigroup. (Kazim and Naseeruddin, (1972)) introduced this concept in 1972. In the ternary commutative identity $klm = mlk$, they presented the parenthesis and introduced the identity:

$$(kl)m = (ml)k, \forall k, l, m \tag{1.1}$$

They name the identity 1.1 as left invertive identity and call the structure left almost semigroup (or shortly LA-Semigroup). Later on, various other names were given to this structure by different other researchers in the literature e.g. "left invertive groupoid (Holgate,(1994)), Abel Grassmann groupoid abbreviated as AG-groupoid (Amanullah, (2014)) and right modular groupoid" (Amayar and Zelinsky,(1966), Amanullah,(2016)). A groupoid or magma can be defined as, an algebraic structure which satisfies the closure property under a binary operation. The term "entropic" and "medial" groupoid were employed by J. Jezek and T. Kepka in their book and cancellative medial, commutative medial, medial division.

Simple medial and quasi groupoids were also studied by them (Hakeem & Ahmad, (2017)). Recently a variety of research has been carried out by various researchers in this area. Fuzzification of the area is also in progress. For the first time "in the Galois Theory of ring" the term groupoid was employed by Amayor and Zelinsky in combinatorial groupoid theory (Naseeruddin, (1970), Distler et al., (2011)). It is easy to prove that each LA-semigroup J is medial that is, it satisfies the medial law:

$(kl)(mn) = (km)(ln) \forall k, l, m, n$ in J (1.2)

It is prominent to mention that in order to avoid the frequent and unnecessary use of parenthesis dots we shall denote the (k ·l)(m·n) as kl·mn and that a·{(b·c)·(d·e)}as a(bc · de).In general LA-semigroups is a non-associative structure that has attracted many researchers for further exploration of this area. As a result, various productive results were explored by these researchers in this new area of research. The structure of LA-semigroups has a variety of applications in different theories like the fuzzy mathematics (Shah et al., (2011)), matrices (Amanullah, (2016)), finite mathematics, flocks' theory (Kazim & Naseeruddin, (1978)) and geometry etc. (Ahmad et al., (20120). Using the modern tools of MACE-4 and GAP, LA- semigroups are enumerated up to order 6 for a variety of subclasses by different researchers [Shah et al., (2012), Shah, (2012), Shah et al., (2011)]. The obtained data is further classified, through which new subclasses are explored (Qaiser, (2004), Mushtaq et al., (1978), Qaiser et al., (2009)]. It is prominent to note that in general LA-semigroup does not require an identity or zero element (Khan, (2008)). However, if it contains a left identity element then it must be unique (Mushtaq et al., (1974), Ahmad & Rauf, (2019)). An LA-semigroup J having a left identity is called an LA-monoid and it always satisfies the paramedial property (Hakeem & Ahmad, (2017):

$$(kl)(mn) = (nl)(mk) \forall k, l, m, n \text{ in } J \qquad (1.3)$$

Amazingly, the structure of LA-semigroup becomes a commutative semigroup if the right identity is allowed in it. Let (Z, +) be the additive Abelian group then one can easily obtain LA-semigroup from the same by introducing a "binary operation ◯" on Z as follows: r∘s = s−r, ∀r, s in Z. For practical example of LA-semigroup, consider the rotation transformations of a square. A square is rotated through 0, 90, 180 and 270 degrees to the right (clockwise) and are denoted by Φe, Φa, Φb, Φc. Denote N =

{Φe, Φa, Φb, Φc}. Obviously, two consecutive rotations have the following results.

$$\Phi e\Phi e = \Phi e, \Phi a\Phi c = \Phi c\Phi a = \Phi e, \Phi b\Phi b = \Phi e.$$

Further, it is easy to show that:

$$\Phi{-1} = \Phi e, \Phi{-1} = \Phi c, \Phi{-1} = \Phi b, \Phi{-1} = \Phi a.$$

e a b c

Now, we define operations ∗ on N as follows:

$$\Phi x * \Phi y = \Phi_x{-1}\Phi y \forall x, y \in \{e, a, b, c\}.$$

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Then, (N, ∗) satisfies the left invertive law, and the operation ∗ is as follows:in the Table. Obviously, (N, ∗) is an LA-semigroup (LA-group).

LA-semigroup (LA-group) generated by transformations of a square

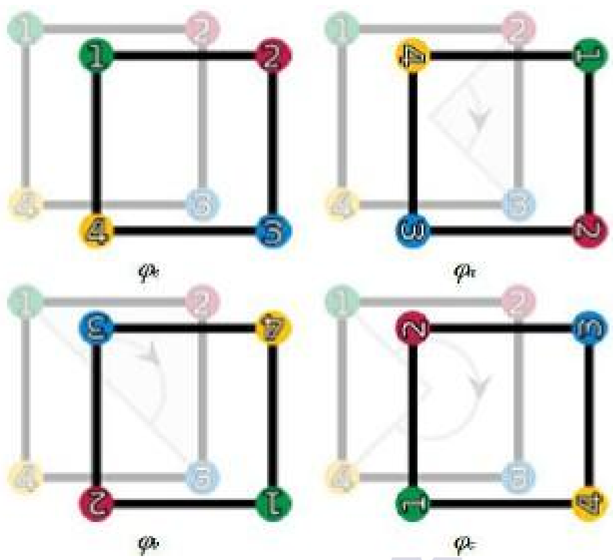

Figure 1.1. Figure 1.1 The rotation of transformations of a square

*	Φe	Φa	Φb	Φc
Φe	Φe	Φa	Φb	Φc
Φa	Φc	Φe	Φa	Φb
Φb	Φb	Φc	Φe	Φa
Φc	Φa	Φb	Φc	Φe

A variety of some new subclasses of LA-semigroup including the following are introduced: "locally associative" [18] satisfying the identity t(t · t) = (t · t)t, ∀t. AG*-groupoid that satisfies any of the two equivalent identities (p · q)r = q(p · r),or (p · q)r = q(r . p), ∀p, q, r. AG**-groupoid [19, 20] with the identity:

p(q · r) = q(p · r)∀p, q, r, and, slim LA-semigroup are introduced, [21, 22] that satisfies the law:

$$p(q \cdot r) = pr, \forall p, q, r.$$

The prominent researchers are strongly inspired, and a lot of research is done in the field after the consideration of these new subclasses. Various researchers studied these subclasses (Qaiser, (2004), Mushatq&Kamran,(1978)), and the relations among these subclasses with other algebraic structures are found in different papers. Compositions and various decom- positions are also introduced for these classes in a variety of papers via different congruences (Cho et al.,(1999), Rashad,(2016)). Ideals in theses subclasses are also defined and investigated (Mushtaq, (2004),Mushatq&Kamran,(1978)). Moreover, fuzzification [25] and soft set theory (Khan, (2011)) for some of these concepts has also been done. The concept of fuzzy and soft set theory has made this structure more attractive for the new researchers [27]. The interaction with this area of research motivated me to study the subclass of "repeated LA-semigroup" that raised in various papers (Ahmed et al.,(2012), Shah et al.,(2012), Shah,(2012), Rashad et al.,(2019)) which satisfies the two identities: $(t\cdot u)(v\cdot w) = (uu)\cdot(vv)\forall t, u, v, w$ (Inner Repeated), $(t\cdot u)(v\cdot w) = (tt)\cdot(ww)\forall t, u, v, w$ (Outer Repeated). The existence for these concepts is depicted by the following non- associative examples of Outer repeated LA-semigroup Inner repeated LA-semigroup and repeated LA-semigroup of order 5,5 and 7 respectively.

*	1	2	3	4	5
1	1	1	1	2	2
2	1	1	1	2	2
3	2	2	2	2	2
4	3	3	3	3	3
5	3	3	3	3	3
Outer Repeated LA					

*	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2
Inner Repeated LA					

*	1	2	3	4	5	6	7
1	2	2	2	2	2	2	2
2	3	3	2	2	2	2	2
3	3	3	2	2	2	2	2
4	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2
7	2	2	2	2	2	2	2
Repeated LA-semigroup							

(Rashad, (2016)) introduced some other classes of LA-semigroups in which one class is known as a slim LA-semigroups. By slim LA-semigroups we mean an LA- semi-group satisfying the identity. The structure of repeated LA-semigroups is not investigated further by any researcher. Proper investigation and detailed examination is needed for this concept. This new notion encouraged me for its detailed study and to put pen to paper about its numerous ideals. We hope that the detailed exploration of this concept shall provide an interesting area for research in algebra and maybe in applied mathematics and medical sciences. It is expected that its effective exploration may open doors to investigate the facts and reasons of inflation and weight gain in human body due to repeated growth of cells.

The concept of slim is introduced by Arif et al. But proper investigation of the same with the repeated LA-semigroup is required. The structure of slim motivated me for detail study in light with the repeated class. This research in combination with the slim- LA-semigroups shall provide a fruitful area in applied sciences on detailed analysis (Jezek &T Kepka, (1983)).

PREMINARIES

In this section we give some basic definitions of various classes of LA-semigroup that arise in a variety of thesis and articles for instance (Kazim & Naseeruddin,(1972), Amanullah,(2016), Distler et al., (2011), Musjtaq et al., (1978), Qaiser & Khan,(2009), Khan,(2008), Ahmed,(2016) and shall be used in the subsequent chapters of this thesis.

Definition 1. (Kazim & Naseeruddin,(1972)), An LA-semigroup J is an algebraic structure that satisfies the left inventive law, (kl)m = (ml)k, ∀k, l, m ∈ J. This structure is also called an AG-groupoid. J is called medial LA-semigroup if it satisfies the medial law i.e.(kl) (mn) = (km)(ln), ∀k, l, m, n ∈ J. It is easy to prove that every LA-semigroup satisfies the medial law [5]. J is called paramedial LA-semigroup if it satisfies the paramedial law [6] i.e.

$$(kl)(mn) = (nl)(mk),\ \forall k, l, m, n \in J.$$

Definition 2. (Mushtaq & Kamran,(1978)) An LA-semigroup J which satisfies the identity, (kl · m)n = (kl·m)(n), ∀k, l, m, n ∈ J is called star left permutable LA-semigroup and J is called star right permutable LA-semigroup if for all k, l, m, n ∈ G, k(lm · n) = m(lk · n). J is called star permutable if it is both star left and star right permutable.

Definition 3. An LA-semigroup J is called —

- **(i)** — slim if it satisfies the slimming property i.e. k(lm) = km, ∀k, l, m ∈ J.
- **(ii)** — J is called Jordon if it satisfies the Jordon law; (kl2)m = (l2)(km), ∀k, l, m ∈J.
- **(iii)** — J is called AG*-groupoid if ∀k, l, m, n ∈ J, (kl)m = l(km) or l(mk), and is called AG**-groupoid if ∀k, l, m, n ∈ J, k(lm) = l(km).
- **(iv)** — An element k of an LA-semigroup J is called idempotent if k2 = k. J is called AG-2-band if all of its elements are idempotent. Moreover, J is called AG-3-band if ∀k ∈ J, (kk)k = k(kk).
- **(v)** — An LA-semigroup J is called bol*-LA-semigroup if ∀k, l, m, n ∈ J it satisfies the identity k(lm · n) = (kl · m)n.

Definition 4. An LA-semigroup J is called —

- **(i)** — anti-commutative LA-semigroup if ∀k, l ∈ J, kl = lk ⇒ k = l.
- **(ii)** — J is called left distributive (LD) LA-semigroup if k·lm = kl·km, ∀k, l, m, n ∈J.

(iii) — J is called right distributive (RD) LA-semigroup if kl · m = km · lm ∀k, l, m, n ∈ J.

(iv) — J is called LAD (left abelian distributive) LA-semigroup if k · lm = kl · mk, ∀k, l, m ∈ J and is called RAD (right abelian distributive) LA-semigroup if kl · m = mk · lm, ∀k, l, m ∈ J.

(v) — J is called bi-commutative if for all ∀k, l, m, n ∈ J it satisfies the identity (kl)(mn) = (nm)(kl).

Definition 5. An LA-semigroup J which also satisfies the right invertive law is called —

(i) — self-dual LA-semigroup, i.e. k(lm) = m(lk), ∀k, l, m, n ∈ J.

(ii) — J is said to be right nuclear square if (kl)m2 = k(lm2), ∀k, l, m, n. J is said to left nuclear square if k2(lm) = (k2l)m, ∀k, l, m, n and is said to be middle nuclear square if (kl2)m = k(l2m), ∀k, l, m, n. J is said to be nuclear square LA-semigroup if it is left, right, middle nuclear square LA-semigroup.

(iii) — J is said to be an LA-monoid, if it contains left identity, ∀k ∈ J, there is an element e in J, such that ek = k.

It may be an easy easy exercise to prove that the left identity in an LA-semigroup if exists is unique and that an LA-semigroup containing right identity becomes a commutative semigroup.

An LA-semigroup J is said to be —

(i) — Stein LA-semigroup if, k(lm) = (lm)k, ∀k, l, m, n ∈ J. Moreover, J is said to be Stein star LA-semigroup if, k(km) = mk, ∀k, l, m, n ∈ J.

(ii) — J is called left Cheban LA-semigroup if then k(lm·n) = mk ·ln, ∀k, l, m, n ∈ J. Similarly, it is called right Cheban LA-semigroup if then (k · lm)n = kn · ml∀k, l, m, n ∈ J. Moreover, J is called Cheban LA-semigroup if it is both left and right Cheban LA-semigroup.

(iii) — J is called left Cheban star LA-semigroup if it satisfies (k(lm)n) = (mk)(ln), ∀k, l, m, n ∈ J and is called right Cheban star LA-semigroup if it satisfies the (k(lm)n) = (mn)(ml), ∀k, l, m, n ∈ J.

(iv) — J is called left (resp. right) cancellative AG- groupoid if it satisfies the identity ku = kv ⇒ u = v, (resp. uk = vk ⇒ u = v), ∀k, u, v ∈ J. Moreover, an LA-semigroup is cancellative if it is left as well as right cancellative LA- semigroup.

(v) — J is known as left (resp. right) alternative LA-semigroup if it satisfies the identity uv · v = u · vv,(resp. uv . v = u . vv) ∀u, v ∈ J.

Definition 6. An LA-semigroup J is known as —

(i) — Moufang LA-semigroup if it satisfies (kl)(mk) = (k(lm))k, ∀k, l, m, n ∈ J.

(ii) — J is known as left Moufang star LA-semigroup if it satisfies (k(lm))n = k(l(mn), ∀k, l, m, n ∈ J. Similarly, J is known as right Moufang star LA- semigroup if it satisfies (k(lm))n = (kl)(mn), ∀k, l, m, n ∈ J.

(iii) — J is known as rectangular LA-semigroup if it satisfies (kl)(km) = (nl)(nm), ∀k, l, m, n ∈ J.

(iv) — J is known as rectangular star LA-semigroup if it satisfies (kl)(mn) = (kn)(ml), ∀k, l, m, n ∈ J.

(v) — J is known as T1 LA-semigroup if it satisfies (kl) = (mn) ⇒ (lk) = (nm), ∀k, l, m, n ∈ J.

(vi) — J is known as T2 LA-semigroup if it satisfies (kl) = (mn) ⇒ (km) = (ln), ∀k, l, m, n ∈ J.

Definition 7. An LA-semigroup J is known as —

(i) — outer repeated (resp. inner repeated) LA-semigroup if it satisfies the Iden- tity (kl)(mn) = (kk)(nn), ∀k, l, m, n ∈ J. (resp. (kl)(mn) = (ll)(mm),∀k, l, m, n ∈ J.)

(ii) — J is known as right repeated if it satisfies the identity (kl)(mn) = (kk)(mm), ∀k, l, m, n ∈J.

(iii) — J is known as left (resp. right) commutative LA-semigroup if it satisfies the identity (kl)m = (lk)m, ∀k, l, m ∈ J. (resp. k(lm) = k(ml), ∀k, l, m ∈ J)

(iv) — J is known as left (resp. right) regular LA-semigroup if it satisfies the identity m · n = m · l ⇒ k · n = k · l, ∀k, l, m ∈ J (resp. m · n = m · l ⇒ m · k = l · m, ∀k, l, m ∈ J).

(v) — J is known as outer (resp. inner) dominant LA-semigroup if it satisfies the identity kl = mn ⇒ kk = nn, ∀k, l, m ∈ J (resp. kl = mn ⇒ l2 = m2, ∀k, l, m ∈ J).

In this Section, we investigate the subclass of repeated LA-semigroup. We start with the following theorem.

Theorem 8. Every repeated LA-semigroup is rectangular star.

$$(k * l) * (m * n) = (k * n) * (m * l)$$

Proof : Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

(kl)(mn) = km · ln by medial law
= kk · nn J is outer repeated
= kn · mn J is outer repeated
= nn · mm J is inner repeated

$= kn \cdot ml$ J is inner repeated

Hence $kl \cdot mn = kn \cdot ml$ and thus J is rectangular star.

Theorem 9. Every repeated LA-semigroup is rectangular.

$$(k * l) * (k * m) = (n * l) * (n * m)$$

Proof: Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

$kl \cdot km = kk \cdot lm$ by medial law
$= kk \cdot ll$ J is inner repeated
$= nk \cdot lm$ J is inner repeated
$= nl \cdot km$ by medial law
$= nl \cdot nm$ J is outer repeated

Theorem 10. Every repeated LA-semigroup is Jordon.

$$k(l2m) = (l2)(km), \forall k, l, m \in J.$$

Proof: Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

$k(l2m) = (ml \cdot l)k$ by left invertive law
$= kl \cdot ml$ by left invertive law
$= ll \cdot mm$ J is inner repeated
$= ll \cdot km$ J is outer repeated
$= l2 \cdot km.$

Theorem 11. Every repeated monoid is semigroup.

Proof: Let J be a repeated LA-semigroup and k, l, m ∈ J. Then

$kl . m = kl . em$ by left identity
$= kk.mm$ J is outer repeated
$= ek.mm$ J is inner repeated
$= ek.lm$ J is outer repeated

= kk.ll J is inner repeated
= ek.lm J is inner repeated
= k · lm by left identity

Theorem 12: Every inner repeated LA-semigroup is paramedial.

Proof: Let J be an inner repeated LA-semigroup, and k, l, m, n ∈ J. Then

kl · mn = ll · mm J is inner repeated

= nl · mk J is inner repeated

The following counter example shows that every repeated LA-semigroup may not be paramedial.

Example 12. Let K = 1, 2, 3, 4. Then (K, ∗) is evidently outer repeated but is not paramedial as (2.1)(1.1)=4.1=3 and (1.1)(1.2)=1.3=1.

*	1	2	3	4
1	1	3	1	1
2	4	4	4	4
3	1	3	1	1
4	3	1	3	3

The following theorem is now appears as a corollary.

Theorem 13. Every repeated LA-semigroup is paramedial.

Proof: Let J be a repeated LA-semigroup, andk, l, m, n ∈ J. Then

kl · mn = kk · nn J is outer repeated
= lk · nm J is inner repeated
= ln · km by medial law
= nn · kk J is inner repeated
= nm · lk J is outer repeated
= nl · mk by medial law

Proposition 14. [MS]: Every paramedial LA–semigroup is left nuclear square.

Using the above result and Theorem 13 the following corollary is obvious.

Corollary 15. Every repeated LA-semigroup is left nuclear square.

Theorem 16. Every repeated LA-semigroup satisfy the identity (kl)(mn) = (mk)(ln).

Proof: Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

$$\begin{aligned} kl \cdot mn &= km \cdot ln && \text{by medial law} \\ &= mm \cdot ll && J \text{ is inner repeated} \\ &= mk \cdot nl && J \text{ is outer repeated} \\ &= kk \cdot nn && J \text{ is inner repeated} \\ &= kk \cdot ln && J \text{ is outer repeated} \\ &= kk \cdot ll && J \text{ is inner repeated} \\ &= mk \cdot ln && J \text{ is inner repeated} \end{aligned}$$

Theorem 17. Every repeated LA-semigroup is biommutative.

Proof. Let J be a repeated LA-semigroup, and k, l, m, n ∈ J.
Then

$$\begin{aligned} (kl)(mn) &= km \cdot ln && \text{by medial law} \\ &= mm \cdot ll && J \text{ is inner repeated} \\ &= km \cdot ln && J \text{ is inner repeated} \\ &= kk \cdot nn && J \text{ is outer repeated} \\ &= lk \cdot nm && J \text{ is inner repeated} \\ &= ln \cdot km && \text{by medial law} \\ &= nn \cdot kk && J \text{ is inner repeated} \\ &= nm \cdot lk && J \text{ is outer repeated} \end{aligned}$$

Theorem 18. Every -repeated LA-semigroup is left permutable.

Proof: Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

$$\begin{aligned} (kl \cdot m)n &= nm \cdot kl && \text{by left invertive law} \\ &= nn \cdot ll && J \text{ is outer repeated} \\ &= mn \cdot lk && J \text{ is inner repeated} \\ &= ml \cdot nk && \text{by medial law} \\ &= ll \cdot nn && J \text{ is inner repeated} \\ &= lm \cdot kn && J \text{ is outer repeated} \\ &= (kn \cdot m)l && \text{by left invertive law} \end{aligned}$$

Hence the result proves.

Proposition 19. Every slim LA-semigroup is bi-commutative.

Proof. Using [46, Theorem 1(i)] .

Theorem 20. A slim LA-semigroup is outer repeated.

Proof: Let J be a repeated LA-semigroup and w, x, y, z ∈ J. Then

$$
\begin{aligned}
xx \cdot zz &= xx \cdot z && \text{J is slim}\\
&= zx \cdot x && \text{by left invertive law}\\
&= zx \cdot yx && \text{J is slim}\\
&= xy \cdot xzby && \text{J is bi-commutative}\\
&= xy \cdot z && \text{J is slim}\\
&= xy \cdot wz && \text{J is slim}
\end{aligned}
$$

Hence J is outer repeated.

Theorem 21. Every repeated with AG** is Bol*.

Proof: Let J be a repeated LA-semigroup, and k, l, m, n ∈ J. Then

$$
\begin{aligned}
k(lm \cdot n) &= lm \cdot kn && \text{J is AG**}\\
&= ll \cdot nn && \text{J is outer repeated}\\
&= kl \cdot nm && \text{J is inner repeated}\\
&= n(kl \cdot m) && \text{J is AG**}\\
&= n(ml \cdot k) && \text{by left invertive law}\\
&= ml \cdot nk && \text{J is AG**}\\
&= mn \cdot lk && \text{by medial law}\\
&= mm \cdot kk && \text{J is outer repeated}\\
&= nm \cdot kl && \text{J is inner repeated}\\
&= (kl \cdot m)n && \text{by left invertive law}\\
\Rightarrow k(lm \cdot n) &= = (kl \cdot m)n
\end{aligned}
$$

Hence the result proved.

Theorem 22. Every outer repeated with AG** is inner repeated.

Proof: Let J be an outer repeated LA-semigroup, and u, x, y, z ∈ J. Then

$$
\begin{aligned}
xy \cdot zu &= xy \cdot uz && \text{J is right commutative}\\
&= u(xy \cdot z) && \text{J is AG**}\\
&= u(z \cdot xy) && \text{J is right commutative}\\
&= u(z \cdot yx) && \text{J is right commutative}\\
&= u(yx \cdot z) && \text{J is right commutative}\\
&= yx \cdot uz && \text{J is AG**}\\
&= yy \cdot zz && \text{J is outer repeated}
\end{aligned}
$$

Theorem 23. Every Inner repeated with LA** is left nuclear square.

Proof: Let J be an inner repeated LA-semigroup, and u, x, y, z ∈ J. Then

$(xx \cdot y)z$	$=$	$(yx \cdot x)z$	by left invertive law
	$=$	$zx \cdot yx$	by left invertive law
	$=$	$y(zx \cdot x)$	J is AG**
	$=$	$zx \cdot yx$	J is AG**
	$=$	$xx \cdot yy$	J is inner repeated
	$=$	$xx \cdot yz$	J is inner repeated

Theorem 25. Every repeated slim LA-semigroup is left nuclear square.

Proof: Let J be repeated LA-semigroup and a, b, c, d ∈ J . Then

$(aa)(bc)$	$=$	$aa \cdot cb$	J is right commutaitve
	$=$	$ac \cdot ab$	by medial property
	$=$	$ac \cdot a(ab)$	J is slim
	$=$	$(ab)(a \cdot ab)$	J is outer repeated
	$=$	$(ab)(ab \cdot a)$	J is right commutaitve
	$=$	$(ab \cdot a)(ba)$	by Theorem 6.16
	$=$	$(ab \cdot a)(c \cdot ba)$	J is slim
	$=$	$(ab \cdot a)(ba \cdot c)$	J is right commutative
	$=$	$(ab \cdot ba)(ac)$	by medial property
	$=$	$(ab \cdot ab)(ac)$	J is right commutaitve
	$=$	$(aa \cdot bb)(ac)$	by medial law
	$=$	$(aa \cdot b)(ac)$	J is slim
	$=$	$(aa \cdot b)c$	J is slim

Theorem 26. Every repeated with slim LA-semigroup is left repeated.
Definition ab.cd=aa.cc

Proof: Let J be repeated LA-semigroup and a, b, c, d ∈ J. Then

$ab \cdot cd$	$=$	$ac \cdot bd$	by medial law
	$=$	$aa \cdot dd$	J is outer repeated
	$=$	$ab \cdot cd$	J is outer repeated
	$=$	$bb \cdot cc$	J is inner repeated

$$= bb\cdot c \quad \text{J is slim}$$
$$= bb\cdot dc \quad \text{J is slim}$$
$$= bb\cdot dd \quad \text{J is inner repeated}$$
$$= ab\cdot dc \quad \text{J is inner repeated}$$
$$= aa\cdot cc \quad \text{J is outer repeated}$$

Theorem 27. Every repeated with Slim LA-semigroup is Star-Right-Cheban •
Defi- nition (a.bc)d=cd.cb

Proof: Let J be repeated LA-semigroup and a, b, c, d ∈ J. Then

$$(a\cdot bc)d = (d\cdot bc)a \quad \text{by left invertive law}$$
$$= dc\cdot a \quad \text{J is slim}$$
$$= dc\cdot aa \quad \text{J is slim}$$
$$= dd\cdot aa \quad \text{J is outer repeated}$$
$$= db\cdot ca \quad \text{J is outer repeated}$$
$$= bb\cdot cc \quad \text{J is inner repeated}$$
$$= ab\cdot cd \quad \text{J is inner repeated}$$
$$= aa\cdot dd \quad \text{J is outer repeated}$$
$$= ac\cdot bd \quad \text{J is outer repeated}$$
$$= cc\cdot bb \quad \text{J is inner repeated}$$
$$= cd\cdot cb \quad \text{J is outer repeated}$$

Theorem 28. Every repeated Slim LA-semigroup is right-regular. •

Definition ab.ac=db.dc

Proof: Let J be repeated LA-semigroup and a, b, c, d, z ∈ J. Then

$$ab\cdot ac = aa\cdot bc \quad \text{by medial law}$$
$$= aa\cdot c \quad \text{J is slim}$$
$$= ca\cdot a \quad \text{by left invertive law}$$
$$= ca\cdot ba \quad \text{J is slim}$$
$$= aa\cdot bb \quad \text{J is inner repeated}$$
$$= ba\cdot bd \quad \text{J is inner repeated}$$
$$= bb\cdot dd \quad \text{J is outer repeated}$$
$$= db\cdot dc \quad \text{J is inner repeated}$$
$$= dd\cdot cc \quad \text{J is outer repeated}$$
$$= db\cdot dc \quad \text{J is outer repeated}$$

Theorem 29. Every repeated with Slim LA-semigroup is left-Regular •

Definition ac.bc=ad.bd

Proof: Let J be repeated LA-semigroup and a, b, c, d ∈ J. Then

$$
\begin{aligned}
ac \cdot bc &= ab \cdot cc && \text{by medial law}\\
&= bb \cdot cc && \text{J is inner repeated}\\
&= db \cdot cb && \text{J is inner repeated}\\
&= db \cdot b && \text{J is slim}\\
&= bb \cdot d && \text{by left invertive law}\\
&= bb \cdot cd && \text{J is slim}\\
&= bb \cdot cc && \text{J is inner repeated}\\
&= db \cdot cb && \text{J is inner repeated}\\
&= dd \cdot bb && \text{J is outer repeated}\\
&= ad \cdot bd && \text{J is inner repeated}
\end{aligned}
$$

Theorem 30. Every repeated with slim LA-semigroup is RAD.

Definition ab.c=ca.bc

Proof: Let J be repeated LA-semigroup and a, b, c, d ∈ J. Then

$$
\begin{aligned}
ab \cdot c &= ab \cdot cc && \text{J is slim}\\
&= ac \cdot bc && \text{by medial law}\\
&= cc \cdot bb && \text{J is inner repeated}\\
&= ac \cdot bb && \text{J is inner repeated}\\
&= aa \cdot bb && \text{J is outer repeated}\\
&= ca \cdot bc && \text{J is inner repeated}
\end{aligned}
$$

Proposition 34. Every AG* is right-commutative and is paramedial.

Using this result we prove the following.

Theorem 35. Every outer repeated with AG* is inner repeated.

Proof: Let J be an outer repeated LA-semigroup and u, x, y, z ∈ J. Then

$$
\begin{aligned}
xy \cdot zu &= y(x \cdot zu) && \text{J is AG*}\\
&= y(zu \cdot x) && \text{J is right commutative}\\
&= (zu)(yx) && \text{J is AG*}\\
&= (zu)(xy) && \text{J is right commutative}\\
&= yu \cdot xz && \text{by paramedial property}\\
&= yy \cdot zz && \text{J is outer repeated}
\end{aligned}
$$

Theorem 36. Every inner repeated with AG* is outer repeated.

Proof: Let J be a repeated LA-semigroup and u, x, y, z ∈ J. Then

$$
\begin{aligned}
xy \cdot zu &= y(x \cdot zu) && \text{J is AG*} \\
&= y(zu \cdot x) && \text{J is right commutative} \\
&= (zu)(yx) && \text{J is AG*} \\
&= (zu)(xy) && \text{J is right commutative} \\
&= zx \cdot uy && \text{by medial property} \\
&= xx \cdot uu && \text{J is inner repeated}
\end{aligned}
$$

Theorem 37. Every repeated with T2 slim LA-semigroup is locally associative.

Proof: Let J be a repeated LA-semigroup that also satisfies the T2properties and a, b, c, d ∈ J. Then

$$
\begin{aligned}
(aa)a &= (aa)(ba) && \text{J is slim} \\
&= aa \cdot bb && \text{J is inner repeated} \\
\Rightarrow a(aa) &= (ba)(aa) && \text{by T 2} \\
&= aa \cdot aa && \text{J is inner repeated} \\
&= aa \cdot a && \text{J is slim}
\end{aligned}
$$

Here we provide some conjectures for consideration by the researchers of the field.

Conjecture 1. A repeated T2LA-semigroup is commutative semigroup.

Conjecture 2. A left transitive outer (inner) repeated LA-semigroup is a commutative semigroup.

Conjecture 3. An inner repeated LA-semigroup S is outer repeated if any of the following hold.

(1) S is cyclic associative (CA) LA-semigroup
(2) S is self-dual LA-semigroup
(3) S is left distributive (LD) LA-semigroup
(4) S is right distributive (RD) LA-semigroup

Conjecture 4. An outer repeated LA-semigroup S is inner repeated if any of the following hold.

(1) S is CA LA-semigroup
(2) S is self-dual LA-semigroup
(3) S is left distributive (LD) LA-semigroup
(4) S is right distributive (RD) LA-semigroup

Conclusion and Future Work

Various properties of repeated LA-semigroups are investigated. A comparative study of repeated LA-semigroups is carried out for some of the subclasses, which include the classes of AG*, AG** and slim LA-semigroups. This may be further extended to other classes like flexible, nuclear square, cyclic associative and left (resp. right) distributive LA-semigroups. Several conjectures are provided for consideration and further investigation by the researchers on the topic of repeated LA-semigroups. In addition to the mentioned conjectures various other concepts of different ideals and fuzzification and the soft sets concepts shall strengthen the field of repeated LA-semigroups.

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